Section 3.1 Inductively Defined Sets

To define a set S *inductively* is to do three things:

Basis: Specify one or more elements of S.
Induction: Specify one or more rules to construct elements of S from existing elements of S.
Closure: Specify that no other elements are in S (always assumed).

Note: The basis elements and the induction rules are called *constructors*.

Example 1. Find an inductive definition for $S = \{3, 16, 29, 42, \ldots\}$.

Solution: *Basis*: $3 \in S$.
Induction: If $x \in S$ then $x + 13 \in S$.

The constructors are 3 and the operation of adding 13. Also, without closure, many sets would satisfy the basis and induction rule. e.g., $3 \in \mathbf{Z}$ and $x \in \mathbf{Z}$ implies $x + 13 \in \mathbf{Z}$.

Example 2. Find an inductive definition for $S = \{3, 4, 5, 8, 9, 12, 16, 17, 20, 24, 33, \ldots\}$.

Solution: To simplify things we might try to "divide and conquer" by writing S as the union of more familiar sets as follows:

$$S = \{3, 5, 9, 17, 33, \ldots\} \cup \{4, 8, 12, 16, 20, 24, \ldots\}.$$

Basis: $3, 4 \in S$.
Induction: If $x \in S$ then (if x is odd then $2x - 1 \in S$ else $x + 4 \in S$).

Example 3. Describe the set S defined inductively as follows:
Basis: $2 \in S$;
Induction: $x \in S$ implies $x \pm 3 \in$ S.

Solution: $S = \{2, 5, 8, 11, \ldots \} \cup \{-1, -4, -7, -10, \ldots \}$.

Example 4. Find an inductive definition for $S = \{\Lambda, ac, aacc, aaaccc, \ldots\} = \{a^n c^n \mid n \in \mathbf{N}\}$.

Solution: *Basis*: $\Lambda \in S$.
Induction: If $x \in S$ then $axc \in S$.

Example 5. Find an inductive definition for $S = \{a^{n+1} b c^n \mid n \in \mathbf{N}\}$.

Solution: *Basis*: $ab \in S$.
Induction: If $x \in S$ then $axc \in S$.

Example 6. Describe the set S defined by: *Basis*: $a, b \in S$
Induction: $x \in S$ implies $f(x) \in S$.

Solution: $S = \{a, f(a), f(f(a)), \ldots\} \cup \{b, f(b), f(f(b)), \ldots\}$, which could also be written as

$$S = \{f^n(a) \mid n \in \mathbf{N}\} \cup \{f^n(b) \mid n \in \mathbf{N}\} = \{f^n(x) \mid x \in \{a, b\} \text{ and } n \in \mathbf{N}\}.$$

Example 7. Describe the set S defined by: *Basis*: $\langle\, 0 \,\rangle \in S$
Induction: $x \in S$ implies $\text{cons}(1, x) \in S$.

Solution: $S = \{\langle\, 0 \,\rangle, \langle\, 1, 0 \,\rangle, \langle 1, 1, 0 \,\rangle, \ldots\}$.

Infix notation

$\text{cons}(h, t) = h :: t$. Associate to the right. e.g., $x :: y :: z = x :: (y :: z)$.

Example 8. Find an inductive definition for $S = \{\langle\,\rangle, \langle\, a, b \,\rangle, \langle a, b, a, b \,\rangle, \ldots\}$.

Solution: *Basis*: $\langle\,\rangle \in S$.
Induction: $x \in S$ implies $a :: b :: x \in S$.

Example 9. Find an inductive definition for $S = \{\langle\,\rangle, \langle\, \langle\,\rangle \,\rangle, \langle\, \langle\, \langle\,\rangle \,\rangle \,\rangle, \ldots\}$.

Solution: *Basis*: $\langle\,\rangle \in S$.
Induction: $x \in S$ implies $x :: \langle\,\rangle \in S$.

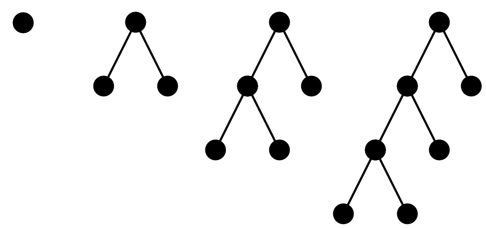
Notation for Binary Trees

Let $t(L, x, R)$ denote the tree with root x, left subtree L, and right subtree R. Let $\langle\ \rangle$ denote the empty binary tree. If $T = t(L, x, R)$, then root(T) = x, left(T) = L, and right(T) = R.

Example 10. Describe the set S defined inductively as follows:

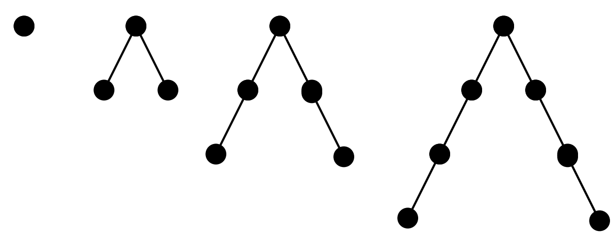
Basis: $t(\langle\ \rangle, \bullet, \langle\ \rangle) \in S$.
Induction: $T \in S$ implies $t(T, \bullet, t(\langle\ \rangle, \bullet, \langle\ \rangle)) \in S$.

Solution (picture): The first few trees constructed from the definition are pictured as follows:



and so on.

Example 11. Find an inductive definition for the set S of binary trees indicated by the following picture.



and so on.

Solution: *Basis*: $t(\langle\ \rangle, \bullet, \langle\ \rangle) \in S$.
Induction: $T \in S$ implies $t(t(\text{left}(T), \bullet, \langle\ \rangle), \bullet, t(\langle\ \rangle, \bullet, \text{right}(T))) \in S$.

Example 12. Find an inductive definition for the set $S = \{a\}^* \times \mathbf{N}$.

Solution: *Basis*: $(\Lambda, 0) \in S$.

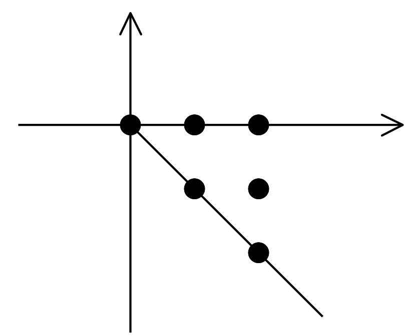
Induction: $(s, n) \in S$ implies $(as, n), (s, n + 1) \in S$.

Example 13. Find an inductive definition for the set $S = \{(x, -y) \mid x, y \in \mathbf{N} \text{ and } x \geq y\}$.

Solution: To get an idea about S we can write out a few tuples:

(0, 0), (1, 0), (1, –1), (2, 0), (2, –1), (2, –2), and so on.

We can also get an idea about S by graphing a few points, as indicated in the picture.



One solution can be written as follows:

Basis: $(0, 0) \in S$.
Induction: $(x, y) \in S$ implies $(x + 1, y), (x + 1, y - 1) \in S$.

Notice that this definition constructs some repeated points. For example, (2, –1) is constructed twice.

Quiz (2 minutes). Try to find a solution that does not construct repeated elements.

Solution: We might use two separate rules. One rule to construct the diagonal points and one rule to construct horizontal lines that start at the diagonal points.

Basis: $(0, 0) \in S$.
Induction: 1. $(x, y) \in S$ implies $(x + 1, y) \in S$.
2. $(x, -x) \in S$ implies $(x + 1, -(x + 1)) \in S$.