Section 2.4 Countability

We can compare the cardinality of two sets by properties of functions.

- $|A| = |B|$ means there is a bijection between A and B.
- $|A| \leq |B|$ means there is an injection from A to B.
- $|A| < |B|$ means $|A| \leq |B|$ and $|A| \neq |B|$.

Example. Let $A = \{3x + 4 \mid x \in \mathbf{N} \text{ and } 0 \leq 3x + 4 \leq 1000\}$. What is $|A|$?

Solution. Notice that $0 \leq 3x + 4 \leq 1000$ iff $-4/3 \leq x \leq 996/3$ iff $-4/3 \leq x \leq 332$. So we can rewrite the definition of $A = \{3x + 4 \mid x \in \mathbf{N} \text{ and } 0 \leq x \leq 332\}$. Now we can define a function $f : \{0, 1, \ldots, 332\} \to A$ by $f(x) = 3x + 4$. Notice that f is a bijection. So $|A| = 333$.

A set S is *countable* if $|S| = |\mathbf{N}_n|$ for some n (i.e., S is finite) OR $|S| = |\mathbf{N}|$. Otherwise S is *uncountable*. If $|S| = |\mathbf{N}|$ we sometimes say that S is countably infinite.

Example. $\mathbf{Z}$ is countable. i.e., $|\mathbf{Z}| = |\mathbf{N}|$. To prove this we need to find a bijection between $\mathbf{Z}$ and $\mathbf{N}$. Let $f : \mathbf{N} \to \mathbf{Z}$ be defined by $f(2k) = k$ and $f(2k + 1) = -(k + 1)$. Check that f is a bijection.

Some Countability Results

1. $S \subseteq \mathbf{N}$ implies S is countable.
2. S is countable iff $|S| \leq |\mathbf{N}|$.
3. Subsets and images of countable sets are countable.
4. $\mathbf{N} \times \mathbf{N}$ is countable. Use Cantor's bijection to associate (x, y) with $((x + y)^2 + 3x + y)/2$.
5. If each of the sets $S_0, S_1, \ldots, S_n, \ldots$ is countable, then so is $S_0 \cup S_1 \cup \ldots \cup S_n \ldots$.

Example. Is $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ countable? Let $S_n = \{(x, y, z) \in \mathbf{N}^3 \mid x + y + z = n\}$. Each of sets $S_0, S_1, \ldots, S_n, \ldots$ is countable (actually finite). So the union $S_0 \cup S_1 \cup \ldots \cup S_n \cup \ldots$ is also countable. But the union is $\mathbf{N}^3$, so $\mathbf{N}^3$ is countable.

Diagonalization Technique

Given a countable listing of sequences $S_0, S_1, \ldots, S_n, \ldots$, where each sequence has the form $S_n = (a_{n0}, a_{n1}, \ldots, a_{nn}, \ldots)$ where each $a_{ni} \in A$ and $| A | \geq 2$. Then there is a sequence over A that is *not* in the listing. A sequence S not in the listing can be defined by choosing two elements $x, y \in A$ and letting $S = (a_0, a_1, \ldots, a_n, \ldots)$ where

$$a_n = \text{if } a_{nn} = x \text{ then } y \text{ else } x.$$

Example. The set of languages over the alphabet $\{a\}$ is uncountable.

Proof: Assume, BWOC, that $L_0, L_1, \ldots, L_n, \ldots$, is a countable listing of all languages over $\{a\}$. We can represent each language L_n by a sequence S_n over $\{0, 1\}$ as follows:

$$S_n = (a_{n0}, a_{n1}, \ldots, a_{nn}, \ldots), \text{ where } a_{ni} = \text{if } a^i \in L_n \text{ then } 1 \text{ else } 0.$$

But now the Diagonlization technique applies to tell us there is a sequence not listed. So there is a language that is not listed. So the set of languages is uncountable. QED.

Cantor's Result: $| A | < | \text{power}(A) |$ for any set A.

Example. Power($\mathbf{N}$) is uncountable because $| \mathbf{N} | < | \text{power}(\mathbf{N}) |$.

Example. The closed interval [0, 1] is uncountable. Show that $| [0, 1] | = | \text{power}(\mathbf{N}) |$.
We can represent each $x \in [0, 1]$ as a binary string $0.b_0b_1b_2\ldots$. Then associate x with the set $S_x = \{i \mid b_i = 1\}$. e.g., 0.10101010… is associated with the set $\{0, 2, 4, 6, \ldots\}$.
This association is a bijection between [0, 1] and power($\mathbf{N}$).