Section 2.2 Constructing Functions

Composition

If $g: A \to B$ and $f: B \to C$, then the expression $f(g(x))$ makes sense and the *composition* of f and g, denoted $f \circ g : A \to C$, is the defined by $(f \circ g)(x) = f(g(x))$.

Some examples: floor($\log_2 20$) = floor(4…) = 4
ceiling($\log_2 20$) = ceiling(4…) = 5
head(tail($\langle a, b, c\rangle$)) = head($\langle b, c\rangle$) = b.

The Sequence and Distribute Functions

seq : $\mathbf{N} \to$ lists($\mathbf{N}$) by seq(n) = $\langle 0, 1, \ldots, n\rangle$.

dist : $A \times$ lists(B) $\to$ lists($A \times B$) by dist(a, $\langle x_1, \ldots x_k\rangle$) = $\langle (a, x_1), \ldots, (a, x_k)\rangle$.

Example. How can we compute $f: \mathbf{N} \to$ lists($\mathbf{N}$) defined by $f(x) = \langle 1, 2, \ldots, x + 1\rangle$?

Solution: Let $f(x)$ = tail(seq(x + 1)).

Quiz (3 minutes). How can we compute $f: \mathbf{N} \to$ lists($\mathbf{N} \times \mathbf{N}$) defined by

$f(x) = \langle (x, 1), (x, 2), \ldots, (x, x)\rangle$?

Solution: Let $f(x)$ = dist(x, tail(seq(x))), or equivalently, tail(dist(x, seq(x))).

Map Function

The *map* function takes a function f and a list and returns the list of values obtained by applying f to each element in the list. So map is defined by

map(f, $\langle x_1, \ldots, x_k\rangle$) = $\langle f(x_1), \ldots, f(x_k)\rangle$.

Quiz (2 minutes). Evaluate the expression map(floor $\circ$ $\log_2$, tail(seq(8))).

Answer: $\langle 0, 1, 1, 2, 2, 2, 2, 3\rangle$.

Example. Find a definition for $f : \mathbf{N} \to \text{lists}(\mathbf{N})$ defined by $f(n) = \langle 0, 2, 4, \ldots, 2n \rangle$.
Solution: Transform the list into a composition of known functions.

$$
\begin{aligned}
f(n) &= \langle 0, 2, 4, \ldots, 2n \rangle \\
&= \langle 2 \cdot 0, 2 \cdot 1, 2 \cdot 2, \ldots, 2 \cdot n \rangle \\
&= \text{map}(\cdot\ , \langle (2, 0), (2, 1), \ldots, (2, n) \rangle) \\
&= \text{map}(\cdot\ , \text{dist}(2, \langle 0, 1, 2, \ldots, n \rangle)) \\
&= \text{map}(\cdot\ , \text{dist}(2, \text{seq}(n))).
\end{aligned}
$$

So define $f(n) = \text{map}(\cdot\ , \text{dist}(2, \text{seq}(n)))$.

Example. Find a definition for $f : \mathbf{N} \to \text{lists}(\mathbf{N})$ defined by $f(n) = \langle 5, 9, 13, \ldots, 5 + 4n \rangle$.

Solution:

$$
\begin{aligned}
f(n) &= \langle 5, 9, 13, \ldots, 5 + 4n \rangle \\
&= \langle 5 + 4 \cdot 0, 5 + 4 \cdot 1, 5 + 4 \cdot 2, \ldots, 5 + 4 \cdot n \rangle \\
&= \text{map}(+, \langle (5, 4 \cdot 0), (5, 4 \cdot 1), (5, 4 \cdot 2), \ldots, (5, 4 \cdot n) \rangle) \\
&= \text{map}(+, \text{dist}(5, \langle 4 \cdot 0, 4 \cdot 1, 4 \cdot 2, \ldots, 4 \cdot n \rangle)) \\
&= \text{map}(+, \text{dist}(5, \text{map}(\cdot\ , \langle (4, 0), (4, 1), (4, 2), \ldots, (4, n\ \rangle)) \\
&= \text{map}(+, \text{dist}(5, \text{map}(\cdot\ , \text{dist}(4, \langle 0, 1, 2, \ldots, n) \rangle)) \\
&= \text{map}(+, \text{dist}(5, \text{map}(\cdot\ , \text{dist}(4, \text{seq}(n))))).
\end{aligned}
$$

So we can define $f(n) = \text{map}(+, \text{dist}(5, \text{map}(\cdot\ , \text{dist}(4, \text{seq}(n)))))$.

Quiz (3 minutes). Find a definition for $f : \mathbf{N}^3 \to \text{lists}(\mathbf{N})$ defined by

$$f(a, b, n) = \langle a, a + b, a + 2b, \ldots, a + nb \rangle.$$

Solution: This is just an abstraction of the previous example, where $a = 5$ and $b = 4$.
So we can define $f(a, b, n) = \text{map}(+, \text{dist}(a, \text{map}(\cdot\ , \text{dist}(b, \text{seq}(n)))))$.