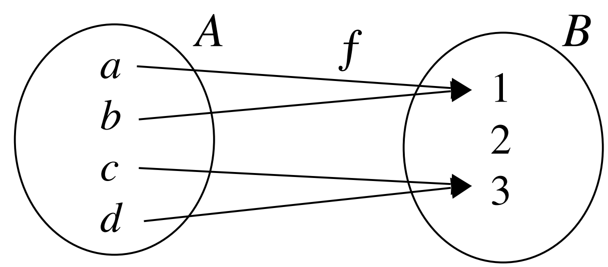
Section 2.1 Functions: Definitions and Examples

A *function* f from A to B associates each element of A with exactly one element of B.
Write $f : A \to B$ and call A the *domain* and B the *codomain*.
Write $f(x) = y$ to mean f associates $x \in A$ with $y \in B$. Say, "f of x is y" or "f maps x to y."

- If $C \subseteq A$, the *image* of C is the set $f(C) = \{f(x) \mid x \in C\}$.
- The *range* of f is the image of A. We write range(f) = $f(A) = \{f(x) \mid x \in A\}$.
- If $D \subseteq B$, the *pre-image* (or *inverse image*) of D is the set $f^{-1}(D) = \{x \mid f(x) \in D\}$.

Example. The picture shows a function $f : A \to B$ with domain $A = \{a, b, c, d\}$ and codomain $B = \{1, 2, 3\}$.



We have: $f(a) = f(b) = 1$ and $f(c) = f(d) = 3$.
Some sample sets are: range(f) = $\{1, 3\}$,
$f(\{a, b\}) = \{1\}$,
$f^{-1}(\{2\}) = \varnothing$,
$f^{-1}(\{1, 2, 3\}) = \{a, b, c, d\}$.

Example. Let $f : \mathbf{Z} \to \mathbf{Z}$ be defined by $f(x) = 2x$. Let E and O be the sets of even and odd integers, respectively. Some sample sets are:

range(f) = $f(\mathbf{Z}) = E$; $f(E) = \{4k \mid k \in \mathbf{Z}\}$; $f(O) = \{4k + 2 \mid k \in \mathbf{Z}\}$;
$f^{-1}(E) = \mathbf{Z}$; $f^{-1}(O) = \varnothing$.

Quiz (2 minutes). Let $f : \mathbf{N} \to \mathbf{N}$ by $f(x)$ = if x is odd then $x + 1$ else x. Find each set, where E and O are the even and odd natural numbers. range(f), $f(E)$, $f(O)$, $f^{-1}(\mathbf{N})$, $f^{-1}(E)$, $f^{-1}(O)$.

Answers: E, E, $E - \{0\}$, $\mathbf{N}$, $\mathbf{N}$, $\varnothing$.

Floor and Ceiling

The *floor* and *ceiling* functions have type $\mathbf{R} \to \mathbf{Z}$, where floor($x$) is the closest integer less than or equal to x and ceiling(x) is the closest integer greater than or equal to x.
e.g., floor(2.6) = 2, floor(–2.1) = –3, ceiling(2.6) = 3, and ceiling(–2.1) = –2.

Notation: $\lfloor x \rfloor$ = floor(x) and $\lceil x \rceil$ = ceiling(x).

There are many interesting properties, all of which are easy to prove.

Example. $-\lfloor x \rfloor = \lceil -x \rceil$.

Proof: If $x \in \mathbf{Z}$, then $-\lfloor x \rfloor = -x = \lceil -x \rceil$. If $x \notin \mathbf{Z}$, there is an $n \in \mathbf{Z}$ such that $n < x < n + 1$, which gives $\lfloor x \rfloor = n$. Multiply the inequality by –1 to get $-n > -x > -(n + 1)$, which gives $\lceil -x \rceil = -n$. So $-\lfloor x \rfloor = -n = \lceil -x \rceil$. QED.

Quiz (1 minute). Show that $\lceil x + 1 \rceil = \lceil x \rceil + 1$.

Proof: There is an integer n such that $n < x \leq n + 1$. Add 1 to get $n + 1 < x + 1 \leq n + 2$.
So $\lceil x + 1 \rceil = n + 2$ and $\lceil x \rceil = n + 1$. Therefore, $\lceil x + 1 \rceil = \lceil x \rceil + 1$. QED.

Greatest Common Divisor (gcd)

If x and y are integers, not both zero, then gcd(x, y) is the largest integer that divides x and y.
For example, gcd(12, 15) = 3, gcd(–12, –8) = 4.

Properties of gcd

- gcd(a, b) = gcd(b, a) = gcd($a, -b$).
- gcd(a, b) = gcd($b, a - bq$) for any integer q.
- gcd(a, b) = $ma + nb$ for some $m, n \in \mathbf{Z}$.
- If $d \mid ab$ and gcd(d, a) = 1, then $d \mid b$.

Division algorithm

For $a, b \in \mathbf{Z}$, $b \neq 0$ there are unique $q, r \in \mathbf{Z}$ such that $a = bq + r$ where $0 \leq r < | b |$.

Euclid's algorithm for finding gcd(a, b)
Assume $a, b \in \mathbf{N}$, not both zero.

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while b > 0 do
        find q, r so that a = bq + r and 0 ≤ r < b;
        a := b;
        b := r
od
Output(a)
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Example. Find gcd(189, 33).

$189 = 33 \cdot 5 + 24$
$33 = 24 \cdot 1 + 9$
$24 = 9 \cdot 2 + 6$
$9 = 6 \cdot 1 + 3$
$6 = 3 \cdot 2 + 0$
Output(3).

Quiz (1 minute). How many loop interations in Euclid's algorithm to find gcd(117, 48)?
Answer: 4 iterations. gcd(117, 48) = 3.

The mod Function

For $a, b \in \mathbf{Z}$ with $b > 0$ apply the division algorithm to get $a = bq + r$ with $0 \leq r < b$. *The remainder r is the value of the mod function applied to a and b*. To get a formula for r in terms of a and b solve the equation for $q = a/b - r/b$. Since $q \in \mathbf{Z}$ and $0 \leq r/b < 1$, it follows that $q = \lfloor a/b \rfloor$. So we have $r = a - bq = a - b \cdot \lfloor a/b \rfloor$.
The value of r is denoted "a mod b". So if $a, b \in \mathbf{Z}$ with $b > 0$ then

$$a \bmod b = a - b \cdot \lfloor a/b \rfloor.$$

Quiz (1 minute). It is 2am in Paris. What time is it in Portland (9 hours difference)?

Answer: (12 hr clock): $(2 - 9) \bmod 12 = (-7) \bmod 12 = -7 - 12\lfloor -7/12 \rfloor = -7 - 12(-1) = 5$.
(24 hr clock) $(2 - 9) \bmod 24 = (-7) \bmod 24 = -7 - 24\lfloor -7/24 \rfloor = -7 - 24(-1) = 17$.

Quiz (1 minute). What are the elements in the set $\{x \bmod 5 \mid x \in \mathbf{Z}\}$?
Answer: $\{0, 1, 2, 3, 4\}$.

Notation: $\mathbf{N}_n = \{0, 1, \ldots, n - 1\}$. So if n is fixed, the range of values for $x \bmod n$ is $\mathbf{N}_n$.

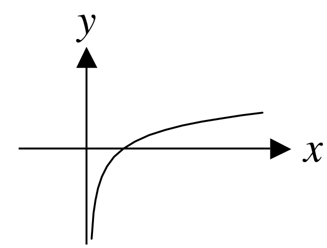
Quiz (1 minute). Convert 13 to binary.
Answer: For any $x \geq 0$ we can write $x = 2\lfloor x/2 \rfloor + x \bmod 2$. So

$$\begin{aligned} 13 &= 2 \cdot 6 + 1 \\ 6 &= 2 \cdot 3 + 0 \\ 3 &= 2 \cdot 1 + 1 \\ 1 &= 2 \cdot 0 + 1 \end{aligned}$$

The remainders from bottom to top are the binary digits from left to right: 1101.

Log Function

If x and b are positive real numbers and $b > 1$, then $\log_b x = y$ means $b^y = x$. The graph of the log function is pictured at the right.



Notice that log is an increasing function: $s < t$ implies $\log_b s < \log_b t$.

Properties of log: $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$,
$\log_b xy = \log_b x + \log_b y$, $\log_b x^y = y \log_b x$, and $\log_a x = (\log_a b)(\log_b x)$.

Quiz (3 minutes). Estimate $\log_2(5^2 2^5)$ by finding upper and lower bounds.

One answer: $\log_2(5^2 2^5) = \log_2(5^2) + \log_2(2^5) = 2\log_2 5 + 5\log_2 2 = 2\log_2 5 + 5$.
Since $4 < 5 < 8$, we can apply $\log_2$ to the inquality to get $2 < \log_2 5 < 3$.
Multiply by 2 to get $4 < 2\log_2 5 < 6$. So $9 < \log_2(5^2 2^5) < 11$.

Another answer: Use $16 < 5^2 < 32$. Then $4 < \log_2(5^2) < 5$. So $9 < \log_2(5^2 2^5) < 10$.