Section 10.2 Boolean Algebra

Motivation: Notice the list of corresponding properties for the algebra of sets and the algebra of propositional wffs.

$\langle$Propositional Wffs; false, true, $\vee$, $\wedge$, $\neg\rangle$	$\langle$power(S); $\varnothing$, S, $\cup$, $\cap$, $'\rangle$
Properties:	Properties:
$A \vee B \equiv B \vee A$	$A \cup B = B \cup A$
$A \vee \text{false} \equiv A$	$A \cup \varnothing = A$
$(A \vee B) \vee C \equiv A \vee (B \vee C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
$A \wedge B \equiv B \wedge A$	$A \cap B = B \cap A$
$A \wedge \text{true} \equiv A$	$A \cap S = A$
$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \wedge \neg A \equiv \text{false}$	$A \cap A' = \varnothing$
$A \vee \neg A \equiv \text{true}$.	$A \cup A' = S$.

These two algebras are concrete examples of a *Boolean algebra* $\langle B; 0, 1, +, \cdot, {}^{-}\rangle$, which has the following properties:

1. + and · are commutative and associative with identity elements 0 and 1, respectively.

2. + and · distribute over each other.

3. $x + \overline{x} = 1$ and $x \cdot \overline{x} = 0$.

Properties of Boolean algebra operations. For any property of the operations there is a *dual property* obtained by interchanging 0 with 1, and + with ·. Similarly, any proof has a *dual proof* obtained in the same way. Here are some basic properties of the operations.

1. $xx = x$ *and* $x + x = x$. Proof: $x = x1 = x(x + \bar{x}) = xx + x\bar{x} = xx + 0 = xx$.
2. $0x = 0$ and $1 + x = 1$. Proof: $0x = (\bar{x}x)x = \bar{x}(xx) = \bar{x}x = 0$.
3. $x + xy = x$ and $x(x + y) = x$. Proof: $x + xy = x1 + xy = x(1 + y) = x1 = x$.
4. $x + \bar{x}y = x + y$ and $x(\bar{x} + y) = xy$. Proof: $x + \bar{x}y = (x + \bar{x})(x + y) = 1(x + y) = x + y$.
5. If $x + y = 1$ and $xy = 0$, then $y = \bar{x}$.
   Proof: $y = y1 = y(x + \bar{x}) = yx + y\bar{x} = 0 + y\bar{x} = x\bar{x} + y\bar{x} = (y + x)\bar{x} = 1\bar{x} = \bar{x}$.
6. $\bar{\bar{x}} = x$. Proof: $\bar{x} + x = 1$ and $\bar{x}x = 0$. So $x = \bar{\bar{x}}$.
7. $\overline{x + y} = \bar{x}\,\bar{y}$ and $\overline{xy} = \bar{x} + \bar{y}$.
   Proof: Show $(x + y) + \bar{x}\,\bar{y} = 1$ and $(x + y)\bar{x}\,\bar{y} = 0$. Then use 5 to obtain result.

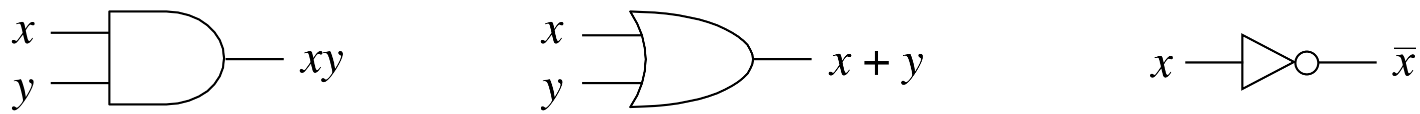
Quiz (2 minutes). Simplify the expression $x + yz + \bar{x}y + \bar{y}xz$.

Answer.

$$
\begin{aligned}
x + yz + \bar{x}y + \bar{y}xz &= (x + x\bar{y}z) + yz + \bar{x}y && \text{(commute and associate)}\\
&= x + yz + \bar{x}y && \text{(absorption)}\\
&= (x + \bar{x}y) + yz && \text{(commute and associate)}\\
&= (x + y) + yz && \text{(absorption)}\\
&= x + (y + yz) = x + y. && \text{(associate and absorption)}
\end{aligned}
$$

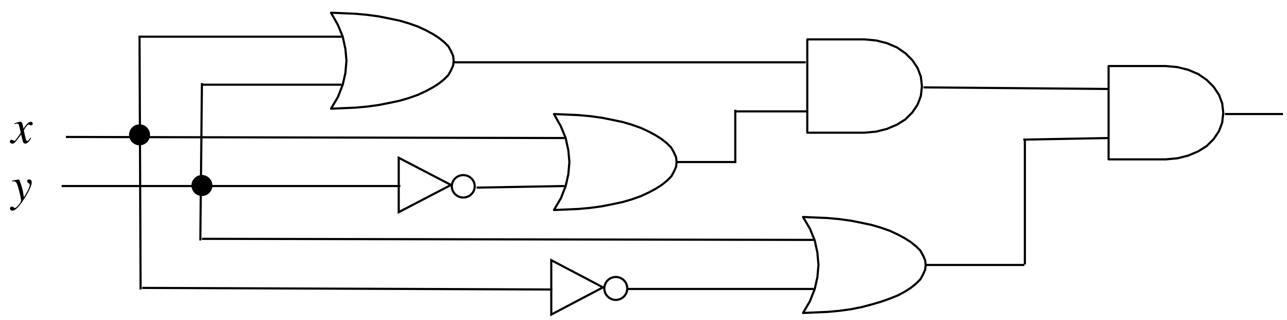
Digital Circuits

A *digital circuit* (or *logic circuit*) is an electronic representation of a truth function. The following three logic gates can be used to implement any digital circuit.



AND gate OR gate NOT gate (inverter)

Quiz. Simplify the following digital circuit.



Solution. The circuit represents the following wff.

$$\begin{aligned}(x+y)(x+\bar{y})(\bar{x}+y) &= [(x+y)x+(x+y)\bar{y}](\bar{x}+y) && \text{(distribute)}\\ &= (x+x\bar{y})(\bar{x}+y) && \text{(absorption, absorption)}\\ &= x(\bar{x}+y) && \text{(absorption)}\\ &= xy. && \text{(absorption)}\end{aligned}$$

So the circuit can be implemented by a single AND gate.