

## Section 1.2 Sets

A **set** is a collection of things.

- If  $S$  is a set and  $x$  is a member or element of  $S$  we write  $x \in S$ . Otherwise we write  $x \notin S$ .
- The set with elements  $x_1, \dots, x_n$  is denoted  $\{x_1, \dots, x_n\}$ .
- The empty set with no elements is denoted  $\{ \}$  or  $\emptyset$ .
- A set with one element is called a singleton. e.g.,  $\{a\}$  is a singleton.
- The set of *integers* is denoted by  $\mathbf{Z}$ , the *natural numbers*  $\{0, 1, \dots\}$  by  $\mathbf{N}$ , the rational numbers by  $\mathbf{Q}$ , and the real numbers by  $\mathbf{R}$ .

### Equal Sets

Two sets  $A$  and  $B$  are **equal**, denoted  $A = B$  if they have the same elements.

e.g.,  $\{a, b, c\} = \{c, b, a\}$  (no ordering).

e.g.,  $\{a, a, b, c\} = \{a, b, c\}$  (no repetitions)

- Sets can be described by properties that the elements satisfy. If  $P$  is a property, then the expression  $\{x \mid P\}$  denotes the set of all  $x$  that satisfy  $P$ .

e.g., The set of odd natural numbers can be represented by the following equal sets.

$$\{x \mid x = 2k + 1 \text{ for some } k \in \mathbf{N}\} = \{1, 3, 5, \dots\}.$$

### Subsets

The set  $A$  is a **subset** of  $B$ , denoted  $A \subset B$ , means every element of  $A$  is an element of  $B$ .

e.g.,  $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$ .

e.g.,  $S \subset S$  for any set  $S$ .

e.g.,  $\emptyset \subset S$  for any set  $S$ .

The **power set** of a set  $S$ , denoted  $\text{power}(S)$  is the set of all subsets of  $S$ .

e.g.,  $\text{power}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ .

*Example(Comparing sets).* Let  $A = \{2k + 7 \mid k \in \mathbf{Z}\}$  and  $B = \{4k + 3 \mid k \in \mathbf{Z}\}$ .

*Quiz.* Is  $A \subset B$ ?

*Answer:* No. For example,  $9 \in A$  but  $9 \notin B$ .

*Quiz.* Is  $B \subset A$ ?

*Answer:* Yes. Let  $x \in B$ . Then  $x = 4k + 3$  for some  $k \in \mathbf{Z}$ . But we can write

$$x = 4k + 3 = 4k - 4 + 7 = 2(2k - 2) + 7.$$

Since  $2k - 2 \in \mathbf{Z}$ , it follows that  $x \in A$ . Therefore  $B \subset A$ . QED.

**Equality in terms of subsets:**  $A = B$  iff  $A \subset B$  and  $B \subset A$ .

*Example.* Let  $A = \{2k + 5 \mid k \in \mathbf{Z}\}$  and  $B = \{2k + 3 \mid k \in \mathbf{Z}\}$ .

Show that  $A = B$ .

*Proof:* First show  $A \subset B$ . Let  $x \in A$ . Then  $x = 2k + 5$  for some  $k \in \mathbf{Z}$ . So we have

$$x = 2k + 5 = 2k + 2 + 3 = 2(k + 1) + 3.$$

Since  $k + 1 \in \mathbf{Z}$ , it follows that  $x \in B$ . Therefore  $A \subset B$ .

Now show the other direction  $B \subset A$ .

.....Class fill in the proof (2 minute quiz).

Since  $A \subset B$  and  $B \subset A$  it follows that  $A = B$ . QED.

## Operations on Sets

Union:  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ .

Intersection:  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .

Difference:  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$ .

Symmetric Difference:  $A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ but not both}\}$

*Note:*  $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ .

Universal Complement: Given a universe  $U$  and  $A \subset U$ , we write

$$A' = U - A.$$

*Example.* For each  $n \in \mathbf{N}$  let  $D_n = \{x \in \mathbf{N} \mid x \text{ divides } n\}$ . So  $D_n$  is the set of positive divisors of  $n$ . Here are some expressions involving these sets.

- $D_0 = \{1, 2, 3, \dots\} = \mathbf{N} - \{0\}$ ,  $D_5 = \{1, 5\}$ ,  $D_6 = \{1, 2, 3, 6\}$ , and  $D_9 = \{1, 3, 9\}$ .
- $D_5 \cup D_6 = \{1, 2, 3, 5, 6\}$
- $D_5 \cap D_6 = \{1\}$
- $D_9 - D_6 = \{9\}$
- $D_5 \oplus D_6 = \{2, 3, 5, 6\}$
- Let  $\mathbf{N}$  be the universe. Then  $D_0' = \mathbf{N} - D_0 = \{0\}$ , and  $\{0\}' = D_0$ .

*Quiz (2 minutes).* Draw a Venn diagram for three sets  $A, B, C$  with some areas shaded. Then find an expression to represent the shaded area.

## Properties of Set Operations

Union and intersection are commutative, associative, and distribute over each other. There are many other properties too. For example,

- Absorption:  $A \cup (A \cap B) = A$  and  $A \cap (A \cup B) = A$ .
- De Morgan's Laws:  $(A \cup B)' = A' \cap B'$  and  $(A \cap B)' = A' \cup B'$ .

## Counting Sets

The *cardinality* of a set  $S$  is denoted by  $|S|$ . Two useful rules for counting finite sets are:

- Inclusion-Exclusion or Union Rule:  $|A \cup B| = |A| + |B| - |A \cap B|$ .
- Difference Rule:  $|A - B| = |A| - |A \cap B|$ .

*Quiz (2 minutes)*. Find a rule for the union of three sets:  $|A \cup B \cup C| = ?$

*Quiz (3 minutes)*. Three programs use a collection of processors in the following way, where  $A$ ,  $B$ , and  $C$  represent the sets of processors used by the three programs:

$$|A| = 20, |B| = 40, |C| = 60, |A \cap B| = 10, |A \cap C| = 8, |B \cap C| = 6.$$

If there are 100 processors available, what could  $|A \cap B \cap C|$  be?

*Answer:*  $100 \geq |A \cup B \cup C| = 20 + 40 + 60 - 10 - 8 - 6 + |A \cap B \cap C|$ . So  $|A \cap B \cap C| \leq 4$ .

**Bags (Multisets)** are like sets but can contain repeated elements. e.g.,  $[t, o, o, t] = [o, t, t, o]$ . Union and intersection can be defined by taking the maximum and minimum occurrences of each element, respectively.

*Quiz (2 minutes)*. Let  $A = [m, i, s, s, i, s, s, i, p, p, i]$  and  $B = [s, i, p, p, i, n, g]$ . What are  $A \cup B$  and  $A \cap B$ ?

*Answer:*  $A \cup B = [m, i, s, s, i, s, s, i, p, p, i, n, g]$  and  $A \cap B = [s, i, p, p, i]$ .