Section 1.1 A Proof Primer

A *proof* is a demonstration that some statement is true. We normally demonstrate proofs by writing English sentences mixed with symbols.

We'll consider statements that are either true or false. If *A* and *B* be are statements, then "not *A*," "*A* and *B*," and "*A* or *B*," are called *negation*, *conjunction*, and *disjunction*, respectively. "not *A*" is opposite in truth value from *A*. "*A* and *B*" is true exactly when both *A* and *B* are true "*A* or *B*" is true except when both *A* and *B* are false.

Conditionals: "if *A* then *B*" (or "*A* implies *B*") is a *conditional* statement with *hypothesis A* and *conclusion B*. It's *contrapositive* is "if not *B* then not *A*" and it's *converse* is "if *B* then *A*". Statements with the same truth table are said to be *equivalent*. The table shows that a conditional and it's contrapositive are equivalent.

A	B	if A then B	if not B then not A
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

A conditional is *vacuously* true if its hypothesis is false.
A conditional is *trivially* true if its conclusion is true.

Proof Techniques: We'll give sample proofs about numbers. Here are some definitions.

- *integers*: …, -2, -1, 0, 1, 2, …
- odd integers: …, -3, -1, 1, 3, … (have the form $2k + 1$ for some integer k).
- even integers:…, -4, -2, 0, 2, 4, … (have the form $2k$ for some integer k).
- $m \mid n$ (read *m divides n*) if $m \neq 0$ and $n = km$ for some integer k.
- p is *prime* if $p > 1$ and its only divisors are 1 and p.

Exhaustive Checking

Some statements can be proven by exhaustively checking a finite number of cases.

Example 1. There is a prime number between 200 and 220.
Proof: Check exhaustively and find that 211 is prime. QED.

Example 2. Each of the numbers 288, 198, and 387 is divisible by 9.
Proof: Check that 9 divides each of the numbers. QED.

Conditional Proof

Most statements we prove are conditionals. We start by assuming the hypothesis is true. Then we try to find a statement that follows from the hypotheis and/or known facts. We continue in this manner until we reach the conclusion.

Example 3. If x is odd and y is even then $x - y$ is odd.
Proof: Assume x is odd and y is even. Then $x = 2k + 1$ and $y = 2m$ for some integers k and m. So we have

$$x - y = 2k + 1 - 2m = 2(k - m) + 1,$$

which is an odd integer since $k - m$ is an integer. QED.

Example 4. If x is odd then x^2 is odd.
Proof: Assume x is odd. Then $x = 2k + 1$ for some integer k. So we have

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is an odd integer since $2k^2 + 2k$ is an integer. QED.

Example 5. If x is even then x^2 is even.
Proof: Class do as *one minute quiz*.

Example 6. If x^2 is odd then x is odd.
Proof: The contrapositive of this statement is "if x is even, then x^2 is even," which is true by Example 5. QED.

Example 7. If x^2 is even then x is even.
Proof: This is the contrapositive of Example 4, which has been shown to be true. QED.

If And Only If (Iff) Proofs

A statement of the form "A if and only if B" means "A implies B" and "B implies A." So there are actually two proofs to give. Sometimes the proofs can be written as a single proof of the form "A iff C iff D iff … iff B," where each iff statement is clear from previous information.

Example 8. x is even if and only if $x^2 - 2x + 1$ is odd.

Proof: x is even iff $x = 2k$ for some integer k	(definition)
iff $x - 1 = 2k - 1$ for some integer k	(algebra)
iff $x - 1 = 2(k - 1) + 1$ for some integer $k - 1$	(algebra)
iff $x - 1$ is odd	(definition)
iff $(x - 1)^2$ is odd	(Examples 4 and 6)
iff $x^2 - 2x + 1$ is odd	(algebra). QED.

Proof By Contradiction

A false statement is called a *contradiction*. For example, "S and not S" is a contradiction for any statement S. A truth table will show us that "if A then B," is equivalent to "A and not B implies false." So to prove "if A then B," it suffices to assume A and also to assume not B, and then argue toward a false statement. This technique is called *proof by contradiction*.

Example 9. If x^2 is odd then x is odd.
Proof: Assume, BWOC, that x^2 is odd and x is even. Then $x = 2k$ for some integer k. So we have

$$x^2 = (2k)^2 = 4k^2 = 2(2k^2),$$

which is even since $2k^2$ is an integer. So we have
x^2 is odd and x^2 is even, a contradiction. So the statement is true. QED.

Example 10. If $2 \mid 5n$ then n is even.
Proof: Assume, BWOC, that $2 \mid 5n$ and n is odd. Since $2 \mid 5n$, we have $5n = 2d$ for some integer d. Since n is odd, we have $n = 2k + 1$ for some integer k. Then we have

$$2d = 5n = 5(2k + 1) = 10k + 5.$$

So $2d = 10k + 5$. Solve for 5 to get

$$5 = 2d - 10k = 2(d - 5k).$$

But this says that 5 is an even number, a contradiction. So the statement is true. QED.