CS 340 SAMPLE EXAM QUESTIONS

P. Let S be the statement,

"If $x + y$ is prime, then x is prime or y is prime."

a. The contrapositive of S is,____

b. The converse of S is,____

P. Use complete sentences to write a proof of the following statement about integers. Use only the definition of "divides" along with algebra.

Prove that if $x \mid y$ and $x \mid z$, then $x \mid (2y + 3z)$.

P. Prove the following statement about the integers.

If x is odd and y is odd, then $x + y$ is even.

P. Evaluate each expression.

a. power($\{\varnothing, a, b\}$) = ____

b. $\{a, b, c, d\} - (\{a, b, c\} \cap \{b, c, d, e\})$ = __________________________________

c. $\{a, b, c\} - (\{a, b, c, d\} - \{c, d, e\})$ = ___________________________________

d. $\{a, b, c\} \oplus \{a, b, c\}$ =____

e. (Let **N** be the universe). $\{1, 2, 3\}' - \{3, 4, 5\}'$=____________________________

f. $\{a, b\} \times \{1, 2, 3\}$ =____

g. head($\langle a, b, c, d\rangle$) = ____

h. tail($\langle a, \mathrm{b}, c, d, e\rangle$) = ____

i. cons($b, \langle c, b, a\rangle$) =____

j. floor(x) $\neq$ ceiling(x) if and only if ____________________________________

k. gcd(236, 112) = ____

l. $(-12) \bmod 7$ =____

m. $\log_2(512)$ =____

P. For each integer n let $A_n = \{x \mid x \in \mathbf{N} \text{ and } x \leq n\}$. Evaluate each of the following expressions.

a. $A_8 \cap A_3$ =____________________________

b. $A_8 \cup A_3$ =____________________________

c. $A_8 - A_3$ =____________________________

d. $A_3 - A_8$ =____________________________

P. Given the following relational database of students, where a tuple consists of (students name, major, number of credits).

$$\text{Students} = \{(x, y, z) \mid x \in \text{Name}, y \in \text{Major}, \text{ and } z \in \text{Credits}\}.$$

Use set notation to describe the set of tuples in Students that represent Freshman (i.e., those with less than 45 credits).

P. Given the following facts about three sets A, B, and C.

$|A \cap (B \cup C)| = 100,$
$|A \cap B| = 70,$
$|A \cap C| = 80,$

Find $|A \cap B \cap C|$.

P. Solve the language equation for L.

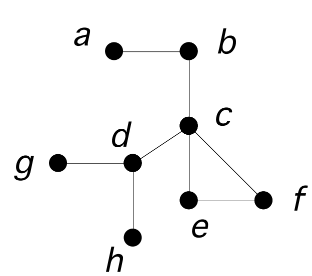
$$\{a, b, ba\}L = \{a, ba, b, ab, bb, bab\}.$$

P. Let L and M be two languages and let $x \in L^*(ML)$. Describe the general form of x by writing it as a concatenation of strings, where each string is in either L or M.

P. Calculate the number of strings over the alphabet $\{a, b, c, d\}$ that have length 7, begin with an a or c, and contain at least one d.

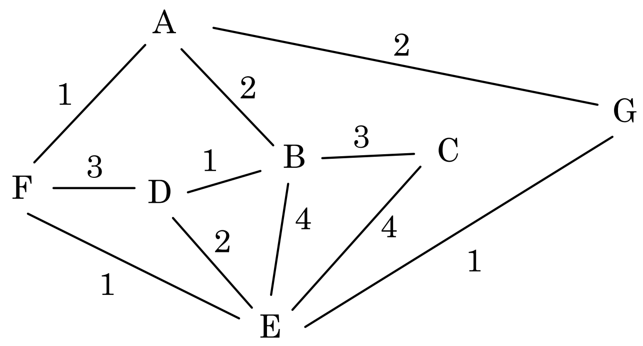
P. Find an expression for the number of strings of length 5 over the alphabet $\{a, b, c, d\}$ that contain at least one d.

P. Given the following graph.



a. Write down a breadth-first search of the graph that starts at c.
b. Write down a depth-first search of the graph that starts at c.

P. Draw a minimal spanning tree for the following weighted graph.



P. Let $f : \mathbf{N}_9 \to \mathbf{N}_9$ be defined by $f(x) = 3x \bmod 9$. Evaluate each expression.

a. $f^{-1}(\{0, 3\})$ = ________________________________

b. $f(\{1, 2, 4\})$ = ________________________________

c. range(f) = ________________________________

P. Evaluate each expression.

a. map(gcd, dist(2, seq(4))).

b. map(f, seq(6)), where $f(x) = x \bmod 5$.

P. Let $f(x) = 3x^2$ and let $g(x, y) = 2x + y$. Find an expression that uses f and g to represent the following expression.

$$6a^2 + 3b^4.$$

P. Express each of the following function definitions as a **composition** of known functions from the set {seq, dist, pairs, map, +, –, *, cons, head, tail}.

a. $f(n, g) = \langle g(0), g(1), ..., g(n)\rangle$.

b. $f(n) = \langle \log_2 (1), \log_2 (2), ..., \log_2(n + 1)\rangle$.

P. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(x) = x \bmod 12$.

a. Show that f is not surjective.

b. Show that f is not injective.

P. Show that the set S of odd integers and the set $\mathbf{Z}$ of integers have the same cardinality. (i.e., find a bijection between the two sets.)

P. Let $f : \mathbf{N}_7 \to \mathbf{N}_7$ be defined by

$$f(x) = (4x + 3) \bmod 7.$$

a. (Fill in the blank.) f is a bijection because gcd(________) = 1.

b. Find a formula for f^{-1}, the inverse of f.

P. Let S = {one, two, three, four, five, six, seven, eight} and suppose that h: $S \to \mathbf{N}_8$ is the hash function defined by

$$h(x) = \text{length}(x) \bmod 8,$$

where length(x) is the number of letters in x. Use h to place each element of S into the following hash table starting with one, then two, and so on until eight. Resolve collisions by **linear probing with a gap of 3.**

0	
1	
2	
3	
4	
5	
6	
7	

P. Given the following countable listing of infinite lists, where each component $x_{ij} \in \{0, 1\}$.

$$\langle x_{00}, x_{01}, x_{02}, \ldots, x_{0n}, \ldots \rangle$$
$$\langle x_{10}, x_{11}, x_{12}, \ldots, x_{1n}, \ldots \rangle$$
$$\langle x_{20}, x_{21}, x_{22}, \ldots, x_{2n}, \ldots \rangle$$
$$\vdots$$
$$\langle x_{n0}, x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots \rangle$$
$$\vdots$$

Describe another infinite list where each component is either 0 or 1 that is not in this listing.

P. Write **countable** or **uncountable** on the line after each set to indicate its cardinality.

a. Rational numbers ____________________

b. Positve real numbers ____________________

c. Negative integers ____________________

d. power(**N**) ____________________

e. $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ ____________________

f. {a, b}* ____________________

P. Write an inductive definition for each set.

a. $S = \{0, 2, 4, ...\}$.

b. $S = \{abc^{n+1} \mid n \in \mathbf{N}\}$.

a. $S = \{a\}^* \times \{b\}^*$. Assume that the basis case is: $(\Lambda, \Lambda) \in S$.

b. $S = \{\langle 1\rangle, \langle 3, 1\rangle, \langle 5, 3, 1\rangle, \langle 7, 5, 3, 1\rangle, \ldots\}$.

P. Write an inductive definition for each set.

a. The set S of odd integers.

b. The set B of binary trees over $\{a\}$ where the left and right subtrees of each node are identical.

P. Show each step in the calculation of $f(47)$, where f is defined by

$f(0) = 0$
$f(n) = f(\text{floor}(n/3)) + n$

P. Write a recursive definition for the following function.

$f(n) = 3 + 5 + ... + (2n + 3)$, where $n \in \mathbf{N}$.

P. Write recursive definitions in each case.

a. For any element x and list L, $f(x, L)$ returns the list obtained from L by removing all occurrences of x from L. e.g., $f(a, \langle b, a, c, a, d\rangle) = \langle b, c, d\rangle$.

b. For any binary tree T, the procedure $p(T)$ prints out only those nodes of T that have two children. The procedure traverses the tree in preorder looking for the nodes to print out.

P. Write a recursive definition for each function or procedure.

a. Let $f : \mathbf{N} \to \text{lists}(\mathbf{N})$ be defined by $f(n) = \langle 2\text{n}, 2(\text{n} - 1), \ldots, 4, 2, 0\rangle$.

b. Let $f : \{a, b, c\}^* \to :\{a, b, c\}^*$ be defined by letting $f(s)$ be the string obtained from s by replacing all occuurences of b by c.

c. For a binary tree T, let *leaves*(T) be a procedure to print out the leaves of T as they occur from left to right.

P. For each of the following relations, write down the properties that the relation satisfies from the list: *reflexive, symmetric, transitive, irreflexive, antisymmetric.*

a. isParentOf, over the set of people.

b. $\neq$, over the set $\mathbf{N}$ of natural numbers.

c. isSubsetOf, over a collection of sets.

P. Given the following binary relations over $\{a, b, c, d\}$.

$$R = \{(a, b), (b, c), (c, c), (d, c)\}$$

$$S = \{(b, a), (c, b), (c, d)\}$$

a. Find $R \circ S$

b. Find $S \circ R$

P. Given the set $A = \{a, b, c, d\}$ and let

$$R = \{(a, b), (b, c), (a, d), (d, b)\}.$$

Draw a directed graph to represent the indicated closure for each of the following binary relations over A.

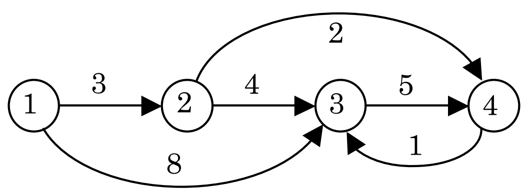
a. $r(R)$

b. $s(R)$

c. $t(R)$

P. Find the transitive closure of $R = \{(1, 2), (3, 1), (3, 2), (2, 4)\}$.

P. Given the following weighted graph.



a. Draw a matrix that can be used to look up the length of shortest paths between any two points.

b. Draw a path matrix that can be used to compute the shortest path between any two points.

P. Find the partition induced by the following equivalence relation over the set **N**.

$$a \sim b \quad \text{iff} \quad a \bmod 4 = b \bmod 4.$$

P. Let $x \sim y$ iff x and y are nonempty lists over $\{a, b\}$ with the same tail.

a. (Fill in the blank). The relation ~ is an equivalence relation because it is the kernel relation of________________________

b. List the elements in each of the following equivalence classes.

$[\langle a \rangle] =$ ________________________________

$[\langle a, b \rangle] =$ ______________________________

$[\langle a, a, b \rangle] =$ ___________________________

P. Let $x \sim y$ iff $x \bmod 3 = y \bmod 3$ over **N**. Describe the partition of **N** caused by ~.

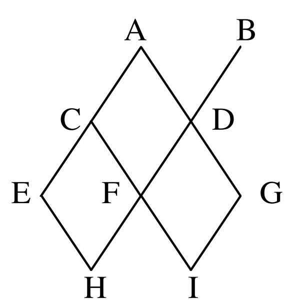
P. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \text{ceiling}(n/4)$. Describe the partition on **N** induced by the kernel relation on f.

P. How would you calculate the smallest equivalence relation that contains a binary relation R?

P. A graph with vertex set {a, b, c, d} has its edges sorted by weight as follows:
{a, b}, {a, c}, {b, c}, {b, d}, {a, d}.

Use Kruskal's algorithm to find a minimal spanning tree T by showing the value of T and the corresponding equivalence classes at each step of the algorithm.

P. Given the following poset diagram for the set {A, B, C, D, E, F, G, H, I}.



Find each of the following items, where S = {C, D, F}.

a. The minimal elements of S.

b. The maximal elements of S.

c. The lower bounds of S.

d. The upper bounds of S

e. The least upper bound of S.

f. The greatest lower bound of S.

P. Let D = {2, 3, 6, 12, 24, 36} and for any x, y ∈ D let $x \prec y$ mean $x \mid y$ (i.e, x divides y). Draw the poset diagram for the partial order $\prec$ on D.

P. Given the poset $\langle \mathbf{N} \times \mathbf{N}, \prec \rangle$, where $(a, b) \prec (c, d)$ means $a + b < c + d$. Write down a descending chain of maximum length that starts with (3, 2).

P. Write an inductive proof that the following statement is true for all natural numbers n.

$$2 + 4 + ... + (2n) = n^2 + n.$$

P. Write out an induction proof of the following equation for all $n \in \mathrm{N}$.

$$3 + 5 + 7 + \ ... + (2n + 3) = (n + 1)(n + 3).$$

P. Let f : lists(**N**) → **N** be defined as follows, where odd(h) tests whether h is an odd number.

$f(\langle\ \rangle) = 0$
$f(h :: T) = \text{if odd}(h) \text{ then } 1 + f(T) \text{ else } f(T).$

Write out an induction proof that, "$f(L)$ is the number of odd numbers in L." for all lists $L \in$ lists(**N**).

P. For any nonempty set S let f : S × lists(S) → **N** be defined as follows:

$f(x, L) =$ if $L = \langle\ \rangle$ then 0
else if $x = \text{head}(L)$ then $1 + f(x, \text{tail}(L))$
else $f(x, \text{tail}(L))$.

Let $P(L)$ = "$f(x, L)$ is the number of occurrences of x in L." Write out an induction proof that $P(L)$ is true for all lists L.

P. Find the unknown quantity for each of the following problems.

a. A solution to a problem has 63 possible outcomes. An algorithm to solve the problem has a ternary decision tree of depth d. What is the smallest value that d could be?

b. A solution to a problem has x possible outcomes. An algorithm to solve the problem has a binary decision tree of depth 5. What can the value of x be?

c. A solution to a problem has 100 possible outcomes. An algorithm to solve the problem has an n-way decision tree of depth 4. What is the smallest value that n could be?

P. Use known closed forms and summation facts to find a closed form for each of the following expressions:

a. $\sum_{i=0}^{n}(3i + 2)$

b. $2\sum_{i=1}^{n} 3^{i+1}$

P. Find an expression for the number of times, in terms of the natural number n, that S is executed in the following algorithm.

```
i := 0;
while i < n do
    j:= i;
    while j < n do S; j := j + 1 od;
    i := i + 1
od
```

P. For each of the following problems, find an expression to represent the answer (DO NOT EVALUATE IT).

a. How many ways can 7 people be arranged in a row.

b. How many ways can 5 people be arranged in a row when the people are chosen from a set of 12 people.

c. How many different sets of 5 cans of soda that can be gotten from a machine that dispenses 4 kinds of soda.

d. How many strings of length 12 over $\{a, b, c, d\}$ contain 2 a's, 2 b's, 5 c's, and 3 d's.

P. Given the following procedure P defined for all natural numbers n.

```
P(n):  if n = 0 then
           C(0)
       else
           C(n);
           P(n – 1)
       fi
```

Let a_n denote the number of times that a certain operation is executed during the execution of $P(n)$. Suppose that $C(n)$ executes the operation $2n$ times. Write down a recurrence to describe a_n. DO NOT SOLVE IT.

P. Given the following recurrence.

$$a_0 = 1.$$
$$a_n = 3a_{n-1} + 4n.$$

Solve the recurrence for a_n by cancellation or substitution. Leave the answer in summation form. You DO NOT have to find a closed form for the answer.

P. Given the following recurrence.

$$a_0 = 1.$$
$$a_1 = 2.$$
$$a_n = 3a_{n-1} + 4a_{n-2}$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Use the generating function technique to find an equation for A(x) that does not involve the summation. STOP when you get to the point where partial fractions are needed.

P. Suppose that we have a recurrence whose nth term is a_n with generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and we calculate that $A(x)$ has the form

$$A(x) = \frac{3}{x+2} + \frac{2}{1-3x}.$$

Find the closed form for a_n.

P. Find truth values for the propositional variables A, B, and C such that the truth value of the following wff is false.

$$(A \vee B \rightarrow C) \wedge A \rightarrow (C \rightarrow B).$$

P. Use basic equivalences to prove the following equivalence.

$$\neg ((\neg A \wedge B) \vee (A \wedge \neg B)) \equiv (\neg A \wedge \neg B) \vee (A \wedge B).$$

P. Use basic equivalences to prove that the following wff is a tautology. In other words, show the wff is equivalent to true.

$$\neg B \wedge (A \rightarrow B) \rightarrow \neg A.$$

P. Demonstrate the use of Quine's method to find out whether the following wff is a tautology, a contradiction, or a contingency.

$$(A \rightarrow B) \rightarrow (A \vee C \rightarrow B \vee C).$$

P. Given the truth function f defined by the following table.

A	B	f(A, B)
true	true	true
true	false	false
false	true	false
false	false	true

a. Write f(A, B) in CNF (conjunctive normal form).

b. Write f(A, B) in DNF (disjunctive normal form).

P. Find each normal form for the following wff. (Full normal form is not required.)

$$(A \to B) \to (C \to D)$$

a. Disjunctive Normal Form (DNF)

b. Conjunctive Normal Form (CNF)

P. Give a formal proof of the following wff using the CP rule. Use only premises and inference rules. i.e., no T's allowed.

$$(A \vee B \to C) \wedge (C \vee D \to E \wedge F) \to (A \to F).$$

P. Find a conjunctive normal form (CNF) for the following wff (Full CNF is NOT required).

$$(A \vee B) \to (C \wedge D).$$

P. Give a formal proof that the following wff is a tautology by using the CP rule. DO NOT use any T's.

$$(A \vee B \to C) \wedge (C \to D \wedge E) \to (A \to D).$$

P. Give a formal proof that the following wff is a tautology by using the CP rule. DO NOT use any T's..

$$(\neg A \vee \neg B) \wedge (B \vee C) \wedge (C \to D) \to (A \to D).$$

P. Give a formal proof that the following wff is a tautology by using the IP rule. Use T only for false.

$$(A \vee B) \wedge (A \to B) \to B.$$

P. Give a formal proof that the following wff is a tautology by using the IP rule. Use T twice: once to transform the negation of the consequent and once for false.

$$(\neg A \vee \neg B) \wedge (B \vee C) \wedge (C \to D) \to (A \to D).$$

P. Write down the proposition denoted by each of the following wffs over the domain $\{a, b\}$.

a. $\forall x \, \exists y \, p(x, y)$.

b. $\exists x \, \forall y \, p(x, y)$

P. In the following wff, label each occurrence of a variable as either **B** (for bound) or **F** (for free).

$$\exists x \, (p(x, y) \wedge \forall y \, (p(x, y, z) \to \exists z \, q(x, y, z, w))).$$

| | | | | | | | | | | | |

P. Describe a model for each of the following wffs.

a. $\exists x\, p(x) \wedge \exists x \neg p(x)$.

b. $\forall x\, \exists y\, (p(x, y) \wedge \neg p(y, x))$.

P. Describe a countermodel for each of the following wffs.

a. $\exists x\, p(x) \rightarrow \forall x\ \ p(x)$.

b. $\exists x\, p(x) \wedge \exists x\ \ q(x) \rightarrow \exists x\ \ (p(x) \wedge q(x))$.

c. $\forall y\, \exists x\ \ p(x, y) \rightarrow \exists x\, \forall y\, p(x, y)$.

P. Find a countermodel for each of the following wffs.

a. $\forall y\, \exists x\, p(x, y) \rightarrow \exists x\, \forall y\, p(x, y)$.

b. $\forall x\, (p(x) \vee q(x)) \rightarrow \forall x\, p(x) \vee \forall x\, q(x)$.

P. Given the following wff.

$$p(a, b) \wedge \forall x\, \forall y\, (p(x, y) \rightarrow q(x, y)) \ \rightarrow\ \exists x\, \exists y\, q(x, y).$$

Prove that the wff is valid as follows: let I be an arbitrary interpretation with domain D such that I is a model for the antecedent. Show that I is a model for the consequent.

P. Given the following wff.

$$p(a, b) \wedge \forall x\, \forall y\, (\neg p(x, y) \vee q(x, y)) \wedge \forall x\, \forall y \neg q(x, y).$$

Prove that the wff is unsatisfiable as follows: Assume that it has a model and find a contradiction.

P. Prove each the following equivalences by starting on one side and proceeding to the other side using known equivalences.

a. $\forall x\, \exists y\, (p(x) \rightarrow q(y)) \ \equiv\ \exists y\ \ \forall x\, (p(x) \rightarrow q(y))$.

b. $\forall x\, (p(x) \rightarrow \forall y\, (q(y) \rightarrow r(x, y))) \equiv \forall x\, \forall y\, (p(x) \wedge q(y) \rightarrow r(x, y))$

P. Use equivalences to construct the indicated prenex normal form for the following wffs. (SHOW ALL WORK)

a. (prenex CNF) $\forall y\, \exists x\, (p(x, y) \wedge \exists x\, q(x)) \rightarrow \forall x\, (p(x, y) \vee q(x))$.

b. (prenex DNF) $\forall x\, (\exists x\, \forall y\, (q(x) \vee r(x, y)) \rightarrow\ p(x))$.

P. Let $C(x)$ mean x is a child, $V(x)$ mean x is a vegetable, and $L(x, y)$ mean x likes y. Find a wff to formalize each of the following sentences about things.

a. Every child hates some vegetable.

b. Some child hates all vegetables.

c. Only adults like vegetables.

P. Give an example a wff $W(x)$ and a term t such that t is not free to replace x in $W(x)$.

P. Write down a formal proof to show the following wff is valid. No T's allowed.

$$\forall x\, (p(x) \to \forall y\, (q(y) \to r(x, y))) \to \forall x\, \forall y\, (p(x) \wedge q(y) \to r(x, y))$$

P. Use the CP rule to give a formal proof of the following wff. Use only premises and inference rules. i.e., no T's allowed.

$$\forall x\, \forall y\, (p(x, y) \to q(x, y) \vee r(x, y)) \wedge \exists x\, \exists y\, (p(x, y) \wedge \neg r(x, y)) \to \exists x\, \exists y\, q(x, y)$$

P. Write down a formal proof using (IP) to show that the following wff is valid. You may use T two times; once to transform the negation of the consequent; once for false.

$$\forall x\, \forall y\, (p(x, y) \to q(x)) \wedge \forall x\, \exists y\, p(x, y) \to \forall x\, q(x).$$

P. Write down a formal proof to show the following wff is valid. No T's allowed.

$$[\exists x\, \forall y\, p(x, y) \wedge \forall x\, (p(x, x) \to \exists y\, q(y, x))] \to \exists y\, \exists x\, q(x, y).$$

P. Prove the correctness of the following wff where x and y are integers.

$$\{x > y + 1\}\ x := x - 1;\ y := y + 1\ \{x \geq y\}.$$

P. Prove the correctness of the following wff, where int(x) means x is an integer.

$$\{\text{int}(x) \wedge x > 0\}\quad \textbf{if } x > 1 \textbf{ then } x := x - 1\ \{x \geq 1\}.$$

P. Prove the correctness of the following wff.

$$\{x = y + z\}\quad \textbf{if } y < z \textbf{ then } z := y \textbf{ else } y := z\quad \{x \geq y + z\}.$$

P. A function f has a *fixed point* x if $f(x) = x$. Write down a formal wff to represent the following sentence.

"The function f has exactly two fixed points."

P. Prove the correctness of the following wff, where x and y take integer values.

$$\{\exists x\, (y = 2x)\}\ y := y - 5\ \{\exists x\, (y = 2x + 1)\}.$$

P. Use the if-then-else rule to prove the correctness of the following wff, where x and y take integer values, and odd(x) and even(x) mean that x is odd and x is even, respectively.

$$\{x < y\}\ \textbf{if}\ \text{odd}(y - x)\ \textbf{then}\ y := y - 1\ \textbf{else}\ x := x + 2\ \{x \le y\}.$$

P. Given the following wff where int(x) means x is an integer, odd(x) means x is odd, and even(x) means x is even.

$$\{(x > 1) \wedge \text{int}(x)\}\ \textbf{while}\ \text{odd}(x)\ \textbf{do}\ x := (x + 1)/2\ \textbf{od}\ \{(x > 1) \wedge \text{int}(x) \wedge \text{even}(x)\}.$$

Use the while-rule to prove the correctness of the wff.

P. Given the following wff where int(x) means x is an integer, odd(x) means x is odd, and even(x) means x is even.

$$\{(x > 1) \wedge \text{int}(x)\}\ \textbf{while}\ \text{odd}(x)\ \textbf{do}\ x := (x + 1)/2\ \textbf{od}\ \{(x > 1) \wedge \text{int}(x) \wedge \text{even}(x)\}.$$

Prove termination of the while-loop, where $(x > 1) \wedge \text{int}(x)$ is the loop invariant.

P. Complete the precondition P so that the following wff matches the Array Assignment Axiom (AAA).

$$\{P\}\ a[n] := t\ \{(a[n] < b) \vee (a[m] \ge c)\}.$$

P. Prove the correctness of the following wff, where $p(x)$ means that x is a positive integer.

$$\{ p(x)\}$$
$$\textbf{while}\ \text{x} > 2\ \textbf{do}\ \text{x} := \text{x} - 2\ \textbf{od}$$
$$\{ p(x) \wedge (x \le 2)\}$$

P. Describe the general process for proving that the following loop terminates, where P is a loop invariant.

$$\{P\}$$
$$\textbf{while}\ C\ \textbf{do}\ S\ \textbf{od}$$
$$\{P \wedge \neg C\}$$

P. Simplify the following Boolean expression.

$$\left(\overline{b} + a\right)\left(a + b\right)$$

P. Perform each of the following Boolean algebra problems.

a. Prove the equality: $\overline{a} + \overline{b} + abc = \overline{a} + \overline{b} + c$.

b. Simplify: $a + b(a + b)$.

c. Simplify: $(a + b) + (\overline{a}\overline{b})$.

d. Simplify: $(a + b)(\overline{b} + a)(\overline{a} + b)$.