Introduction to Nonlinear Structural Problems

Many real world problems involve nonlinear behavior. The underlying principle of nonlinear behavior is that the cause and effect relationships, unlike linear systems, are not proportional. Some examples of nonlinear real world examples include staples, tire deflection under varying load (causing variation in the contact surface), metal forming resulting in permanent plasticity, large deflection of fishing poles, etc. A typical load-deflection curve for nonlinear behavior is shown below. The example shows a staple in original and final (deformed) shape, and the corresponding load vs. deflection curve. This example also illustrates a case of material nonlinearity.

Solution to non-linear problems are not as well developed, and easily understood, as that of linear problems. From the standpoint of the finite element modeling, the familiar equation of equilibrium we studied in the linear solution \((K \mathbf{u} = \mathbf{F})\) is no longer valid. Furthermore, the solution procedure may involve a large number of piecewise linear solutions to approximate the nonlinear behavior.

In nonlinear problems, stiffness matrix does not necessarily remain constant. Depending on the type of nonlinearity, stiffness matrix will have to be updated many times during the solution procedure. The process of updating the stiffness matrix will result in significant increase in solution time compared with linear analysis.

Nonlinear behavior can be grouped into three general behaviors:

- Material Nonlinearity
- Geometric Nonlinearity
- State change

In the following section a brief description of each type of nonlinearity is presented. Finally, a discussion of iterative solution approach to nonlinear problems is presented.
Material Nonlinearity

This type of nonlinearity arises when the material exhibits non-linear stress-strain relationship. Recall that for linear elastic FE analysis the only stress-strain relationship was defined via modulus of elasticity, E. Now, in the case of non-linear material analysis, the modulus of elasticity is only the first definition point of an overall behavior. The typical definition and analysis in the non-linear material domain involves one of post-yield (plastic) behavior. Typical elasto-plastic material characteristic under tension is shown in figure 1. The unloading line determines the residual (plastic) strain remaining in the system.

Note that figure 1 also represents a structure which exhibits a softening behavior after yielding. The numerical solution of this type of non-linear problem involves approximating the non-linear segment of stress-strain curve with a series of piece-wise linear segments.

Each linear segment is approximated by a tangent modulus ($E_t$) which is computed as the ratio of stress over strain for that particular line segment (as shown in figure 2). Upon unloading, the structure retains the permanent (plastic) deformation and the unloading line (as shown in the figure) will be one of linear elastic regime with elastic

Figure 3 represents the typical stress-strain behavior of a structure which hardens with increasing stress value. Here the iterative time step of the solution must be carefully undertaken, as large step size can completely miss the actual stress-strain curve leading to a solution process that will never converge.
For a complete FE analysis three important concepts must be well understood. The first one is the **Yield Criterion**. It determines how the applied stresses (on the component) are related to the yield strength specified in FE study, and determines the onset of yielding. The most commonly used criterion is the *von Mises* (or octahedral shear) theory. Therefore, when the von Mises stress reaches the yield strength, the component is assumed to have yielded, and the plastic regime begins. Other yield criteria include *Tresca* (Maximum shear stress theory) and *Drucker-Prager* (also known as Coulomb-Mohr theory).

The second concept relates the progression of yielding in the plastic domain. This is referred to as the **flow rule**. The most commonly used is the *Prandtl-Reuss* relation which relates the strain increments to the stress increments of the common metals, primarily in elastic-perfectly plastic characteristic.

The final concept describes the mechanism for the growth of the yield surface. It is called the **hardening rule**. It determines how the yield point changes as a result of accumulation of plastic strain, and depends on the type of material. We sometime refer to this as *strain-hardening*. Metals are usually in the category described by *kinematic hardening*. There are a variety of kinematic hardening rules (see, for example, I-DEAS hardening rule section). The *isotropic hardening* rule assumes the center of yield region remains stationary in the stress space while the size of yield surface expands as a result of strain hardening. This theory is best suited for problems in which the plastic strain is considerably more than the onset of yield, such as manufacturing processes (forming, cold working) and large motion dynamic problems.
Geometric Nonlinear analysis

Geometric nonlinearity arises when a system undergoes deformations large enough to change the way the structure resists the load, or the way load is applied to the structure. While there are many situations in which large deformation are accompanied by plastic deformation, the following discussion assumes the large deformations remain within the elastic range, with the inclusion of special geometrically non-linear case known as “buckling.”

The major aspect of geometric non-linearity is the coupling between bending (out-of-plane) and membrane (in-plane) stiffness terms. As for the solution process, as long as the load-response curve remains reasonably continuous (see blow), general solution procedures for nonlinear problems may be applied. However, several points need to be considered.

If the load behavior during the analysis changes, the computer program needs to do so during the iterative solution process. There are usually options available to instruct the program to maintain the original load direction or update it based on the displacement. Furthermore, changes in the system response such as from softening to hardening, or the points of instability, can cause difficulties for the usual incremental load solution. In the figure below, the response on the left is one of change from softening to hardening which includes an “inflection” point. The figure on the right represents a post-buckling phenomenon. Point A is the “limit” point, where the tangent stiffness is zero. However, the structure is not in a condition of equilibrium! Buckling may not occur at A but the structure may undergo a sudden “snap through” to point B.

In both cases the typical solution method based on “load increment” (employed in material nonlinearity) will break down.

*I-DEAS note:* Loads applied to vertices and nodes, as well as surface tractions maintain their original directions. Surface pressures and loads applied to the element free edges (or faces) update with the deformation. Mid-section beam loads update with the deformations if applied in the element coordinate system.
Buckling Analysis

Buckling describes a situation when a structure becomes unstable and may fail without warning.

**Contributing factor:** Slender beams and thin shells can develop “membrane” strain energy when subjected to in-plane loading.

**Compressive loading:**

In-plane stiffness opposes the bending stiffness. At some point the resultant becomes zero, leading to the structure offering no resistance to the applied force. Hence instability (buckling) occurs.

**Tensile loading:**

In-plane stiffness adds to the bending stiffness, and the structure becomes stiffer as the load increases. This phenomenon is known as “stress stiffening.”

Solution of buckling problem is similar to the general eigen-value problem:

\[( [K_0] + \lambda_i [K_\sigma] ) \{ \phi_i \} = \{0\} \]

Where \([K_0]\) is the original structural stiffness matrix based on initial geometry and linear material properties. \([K_\sigma]\) is known as stress stiffness (or geometric stiffness) matrix, resulting from the applied load and restraints. \(\lambda_i\) is the \(i\)-th eigen value, or buckling load factor (or multiplier). \(\{ \phi_i \} \) is the eigenvector corresponding to the buckled mode shapes associated with the buckling load factor \(\lambda_i\). Buckling load is determined by extracting the value of \(\lambda_i\) and multiplying it by the applied load. Therefore, if the actual load applied to the structure during finite element analysis is designated as \(f_i\):

\[\text{Buckling load} = f_i \cdot \lambda_i\]

**Characteristics of Linear Buckling**

- Stiffness terms are based on the **undeformed** geometry.
- Membrane stresses do not change in distribution.
- Often overestimates the actual buckling load.

Works well for straight columns and flat-plates, which are assumed to remain free of bending until buckling occurs.

**Note:**

In most thin-walled structures, membrane and bending stresses will develop simultaneously, and coupling of the two terms may develop well before buckling occurs. The interaction may change the membrane stress distribution, causing it to vary non-linearly with the load. Also, stiffness can become a function of displacements if displacements are large or yielding is present.
Non-linear Solution approaches

The objective of non-linear analysis solution is to solve the set of familiar equations $[k]{u} = {F}$. The difference is that $[k]$ itself is a function of the displacement, written as $k(u)$. Some solution approaches are better suited for certain classes of non-linearity while others are best used for different non-linear systems. We describe several general approaches to non-linear problem solving.

The two most common approaches are incremental “predictor” and the iterative “corrector” methods.

Figure 1 shows the predictor method. In this approach each solution step is approximated by a line segment. It can be shown that at each step the solution diverges a little more from the correct one, and thus this method is susceptible to the error build-up. This method is also known as a forward approach.

![Fig. 1 Predictor method of nonlinear solution approximation.](image)

To correct the situation, the corrector method is usually employed. Two variations of this include:

1) Newton-Raphson
2) Modified Newton-Raphson
The most common implementation of the N-R approach combines the incremental and the iterative solutions. In the case shown in figure 2, the tangential solution can usually be used as a starting point (predictor) for the solution process, followed by subsequent iterations based on newly computed stiffness matrices. In the Newton Raphson approach, a load vector “imbalance” is calculated upon completion of each iteration step. This imbalance vector refers to the difference between the forces corresponding to the element stresses and the applied load. The computer program performs a linear solution based on the imbalance load vector, and checks for convergence.

Fig. 2 Newton-Raphson approximation

Fig. 3 Modified Newton-Raphson approximation

If convergence is not achieved, the imbalance vector is recomputed, followed by the updating of the stiffness matrix. The FE set of equations are solved again to obtain a new solution. This iterative process is continued until convergence has been achieved, or a determination made that the problem cannot be adequately solved. The Newton-Raphson approach can become very expensive in large DOF problems due to the fact the tangent stiffness matrix \([k]\) must be computed for each iteration step.

The alternative method is known as the Modified Newton-Raphson approach (shown in fig. 3). In this method the tangent stiffness matrix is not updated, and iterations at each load step are performed using the same initial stiffness matrix. So, for any given load level, the tangent stiffness matrix is computed only once. While this reduces the number of stiffness matrix computations, the number of equilibrium iterations to achieve convergence increases considerably. Note the above examples are shown for softening structures. The equilibrium iteration and the choice of initial tangent stiffness are more critical for hardening structure. A wrong first choice may lead to a divergent solution irrespective of subsequent attempt.

The foregoing discussion referred to procedures that are commonly referred to as “Newton” method. There exits a body of approaches referred to as “quasi-Newton” methods. These techniques call for the updating of the inverse of the stiffness matrix rather than the stiffness matrix itself. This technique seems to greatly reduce the computation time. The methods for updating the
inverse matrix during the iterative solution process are well known and widely employed in the design optimization and nonlinear search algorithms. One of the most widely used is the BFGS (Brydon-Fletcher-Goldfarb-Shanno) search method.

Overall, the nonlinear solution technique is a collection of piece-wise linear solves, with each step representing one increment of load. During each load step an iterative approach is employed to make sure the structure has reached equilibrium, or the solution has converged. A number of convergence-enhancement and recovery features, such as line search, automatic load stepping, and bisection, can be activated to help the problem to converge. If convergence cannot be achieved, then the program attempts to solve with a smaller load increment.

In some nonlinear static analyses, if you use the Newton-Raphson method alone, the tangent stiffness matrix may become singular (or non-unique), causing severe convergence difficulties. Such occurrences include nonlinear buckling analyses in which the structure either collapses completely or "snaps through" to another stable configuration. For such situations, you can activate an alternative iteration scheme, the arc-length method, to help avoid bifurcation points and track unloading. The arc-length method causes the Newton-Raphson equilibrium iterations to converge along an arc, thereby often preventing divergence, even when the slope of the load vs. deflection curve becomes zero or negative. Therefore, this method (or some variation of it) is commonly employed in the nonlinear post-buckling analysis.
Convergence

At each load step, the equilibrium iterations are said to converge when the solution is deemed to be “close enough.” How this is defined is entirely up to the analyst and the nature of problem being addresses. A typical convergence criterion is based on the Euclidean norm error defined as:

\[
\text{error} = \frac{\| e_F \|}{\| R_i \|}
\]

where \( \| e_F \| \) is the SRSS (square root of the sum of the squares) of the current iteration load imbalances for all DOF’s and \( \| R_i \| \) is the SRSS of the current load applied to all DOF’s. Note that in case of a small error value, say \( \text{error} = 0.01 \) the iteration may be terminated and the current iteration is said to have converged. The same ratio can be defined for quantities such as displacements, strain energy norms, etc. One note of caution is that in every solution approach, there is an iteration “counter” that may not be exceeded. Reaching a limit number of iteration before achieving convergence is sometimes referred to as “convergence failure.” The remedy for this is for the iteration to restart with different initial conditions such as load increment, stiffness value, etc. Usual computer software remedies involve automatic restart of the solution from the previous load level but using a smaller load step (such as dividing the “convergence failure” load step by half).

References:
