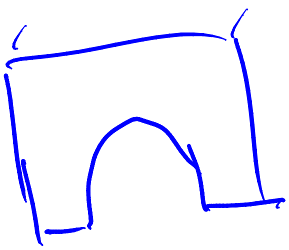
2-D ELEMENTS:

PLANAR ANALYSIS: GEOMETRY, LOADS, B.C's AND RESULTING STRESS & DEFORMATIONS OCCUR IN-PLANE ONLY.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [E] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$[E]$ — CONSTITUTIVE MATRIX



$[E]$ DEPENDS ON TWO CLASSIC PROBLEMS:

1) PLANE STRESS

2) PLANE STRAIN

$E \equiv$ MODULUS OF ELASTICITY

$\nu \equiv$ POISSON'S RATIO

$$[E] = \frac{E}{(1+\nu)(1-\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

PLANE STRESS

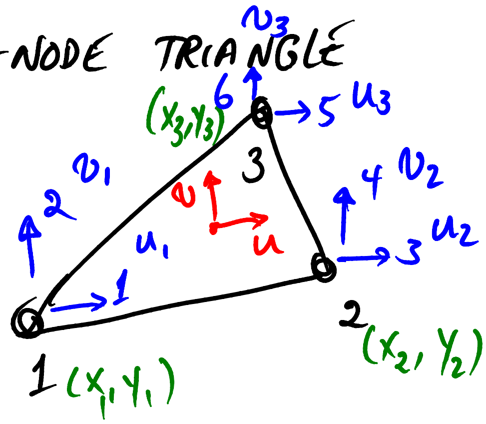
$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

PLANE STRAIN

SIMPLEST 2-D ELEMENT SHAPE IS A

TRIANGLE:

SIMPLEST $\rightarrow$ 3-NODE TRIANGLE



$$\begin{cases} u(x,y) = a_1 + a_2 x + a_3 y \\ v(x,y) = a_4 + a_5 x + a_6 y \end{cases}$$

ASSUMED DISP. FUNCTIONS

(DESCRIBE DISP. BEHAVIOR) AT ANY POINT WITHIN ELEMENT.

$$\epsilon_x = \frac{\partial u}{\partial x} \quad \epsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$U = \int_{vol} \frac{1}{2} \{\sigma\}^T \{\epsilon\} dvol.$$

APPLY B.C.'s FOR ELEMENT:

$$\begin{matrix} x = x_1 \\ y = y_1 \end{matrix} \Rightarrow \begin{cases} u = u_1 \\ v = v_1 \end{cases} \Rightarrow \begin{matrix} a_1 + a_2 x_1 + a_3 y_1 = u_1 & (1) \\ a_4 + a_5 x_1 + a_6 y_1 = v_1 & (2) \end{matrix}$$

$$a_4 + a_5 x_3 + a_6 y_3 = v_3 \quad (6)$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\{a\} = [X]^{-1} \{u\}$$

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$\alpha_1 = x_2 y_3 - y_2 x_3 \qquad \alpha_2 = y_1 x_3 - x_1 y_3 \qquad \alpha_3 = x_1 y_2 - y_1 x_2$

$\beta_1 = y_2 - y_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2 \qquad \gamma_2 = x_1 - x_3 \qquad \gamma_3 = x_2 - x_1$

$$2A = \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} [\alpha] \\ [\beta] \\ [\gamma] \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} [\alpha] \\ [\beta] \\ [\gamma] \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

EXPRESSING DISP. VECTOR $\{u\}$ IN TERMS OF COEFF.'s a_i's:

$$\{u\} = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

SUBST. FOR $\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$:

$$\{u\} = \frac{1}{2A} [1 \quad x \quad y] \begin{bmatrix} [\alpha] \\ [\beta] \\ [\gamma] \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\{u\} = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

↳ SHAPE FUNCTION TERM

$$\{v\} = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$N_1 = \frac{1}{2A} (\alpha_1 + \beta_1 x + \gamma_1 y)$$

$$N_2 = \frac{1}{2A} (\alpha_2 + \beta_2 x + \gamma_2 y)$$

$$N_3 = \frac{1}{2A} (\alpha_3 + \beta_3 x + \gamma_3 y)$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{u\} = [N] \{d\}$$

$$N_3 = \frac{1}{2A}\left(\alpha_3 + \beta_3 x + \gamma_3 y\right)$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\underbrace{\{u\}}_{\text{DISP. FIELD WITHIN ELEM.}} = \underbrace{[N]}_{\text{SHAPE FUNCTION MATRIX}} \underbrace{\{d\}}_{\text{VECTOR OF NODAL DISP'S}}$$

STRAIN:

$$\{\epsilon\} = [B]\{d\}$$

$[B]$ – DERIVATIVE OF $[N]$ – STRAIN-DISP. MATRIX

$$\begin{Bmatrix} \epsilon_x = \dfrac{\partial u}{\partial x} \\ \epsilon_y = \dfrac{\partial v}{\partial y} \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \boxed{\begin{bmatrix} Y_{23} & 0 & Y_{31} & 0 & Y_{12} & 0 \\ 0 & X_{32} & 0 & X_{13} & 0 & X_{21} \\ X_{32} & Y_{23} & X_{13} & Y_{31} & X_{21} & Y_{12} \end{bmatrix}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$X_{ij} = X_i - X_j \qquad Y_{ij} = Y_i - Y_j$$

$[B]$ CONSTANT TERMS

$$A = \text{AREA OF ELEM (TRIANGLE)} = \frac{1}{2}\det[J]$$

$$[J] = \begin{bmatrix} X_{13} & Y_{13} \\ X_{23} & Y_{23} \end{bmatrix} = \text{JACOBIAN MATRIX}$$

DERIVE STIFFNESS MATRIX (POTENTIAL ENERGY)

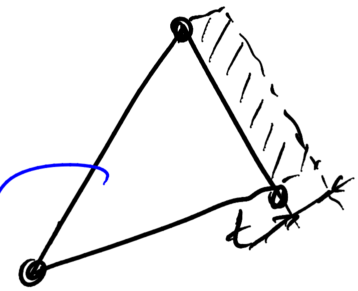
$$\Pi_{TOTAL} = \underbrace{U_{STRAIN}}_{\text{STRAIN ENERGY}} + \underbrace{\Omega_b}_{\text{VOLUM. (BODY) FORCE}} + \underbrace{\Omega_s}_{\text{SURFACE TRACTION}} + \underbrace{\Omega_P}_{\text{CONCENTRATED (POINT) LOADS}}$$

$$U = \frac{1}{2} \iiint_{vol} \{\sigma\}^T \{\epsilon\} \, d\,vol.$$

$$\{\sigma\} = [E]\{\epsilon\}$$

$$\{\sigma\}^T = \{\epsilon\}^T [E]^T$$

$$U = \frac{1}{2} \iiint_{vol.} \{\epsilon\}^T [E]^T \{\epsilon\} \, d\,vol.$$



$$\{\epsilon\} = [B]\{d\} \qquad d\,vol = t \cdot dA$$

$$\{\epsilon\}^T = \{d\}^T [B]^T$$

$t \equiv$ ELEM. THICKNESS

$$U = \frac{1}{2} \int_{AREA} \underbrace{\{d\}^T [B]^T}_{\{\epsilon\}^T} [E]^T \underbrace{[B]\{d\}}_{\{\epsilon\}} \underbrace{t \cdot dA}_{d\,vol}$$

= FINITE (PLANE STRESS)

= 1 (UNITY) (PLANE STRAIN)

$$U = \frac{1}{2} \{d\}^T \left(\int_{AREA} [B]^T [E]^T [B] \, t \cdot dA \right) \{d\}$$

COMARE

$$u = \frac{1}{2} \{x\}^T \; [K] \; \{x\}$$

STIFFNESS MATRIX : $[k] = \int_{AREA} [B]^T [E]^T [B] \, t \cdot dA$

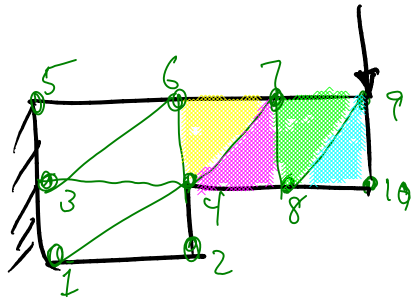
$[B]^T$, $[B]$, $[E]^T$ ARE CONST. (NOT FUNCTIONS OF x OR y)

THUS :

$$[k] = [B]^T [E]^T [B] \, t \int_{AREA} dA$$

$$\boxed{[k] = [B]^T [\epsilon]^T [B] \, t \cdot A}$$

PHYSICAL PROPERTY NEEDS TO BE SPECIFIED IS THICKNESS!



TOTAL No. OF D.O.F.'s $= 20$

6 D.O.F.'s WILL BE ELIMINATED UPON APPLICATION OF CONSTRAINTS

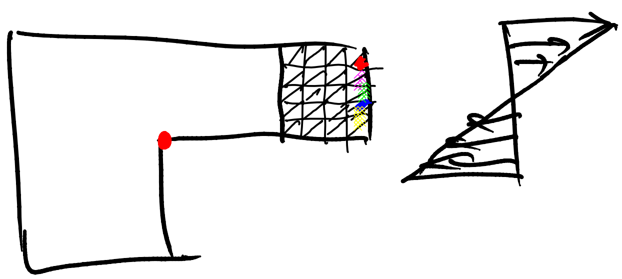
$$u = a_1 + a_2 x + a_3 y$$

$$v = a_4 + a_5 x + a_6 y$$

$$\epsilon_x = \frac{\partial u}{\partial x} = a_2 \qquad \epsilon_y = \frac{\partial v}{\partial y} = a_6 \qquad \rightarrow \text{CONST.}$$

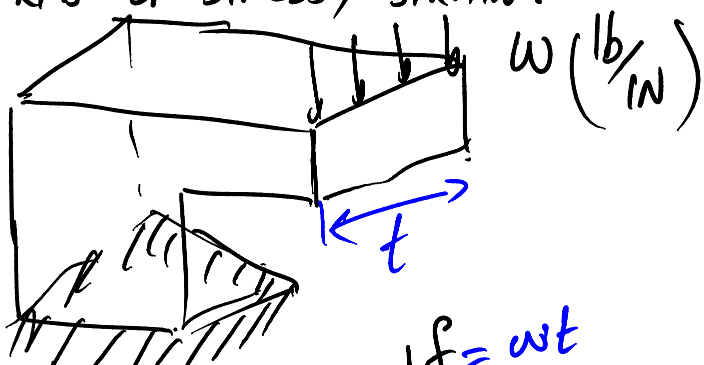
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + a_5$$

CONSTANT STRAIN TRIANGLE (CST) ELEMENT



$\sigma = \frac{MC}{I}$

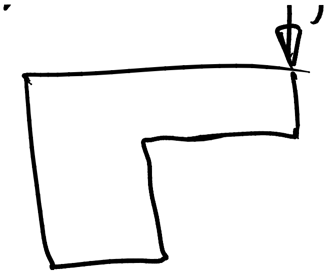
TO CAPTURE BENDING (OR ANY OTHER NON-CONST. STRESS DISTRIBUTION) ONE MUST USE MESH REFINEMENT TO FORCE CST TO REVEAL PROPER DISTRIB. OF STRESS / STRAIN.



MODEL

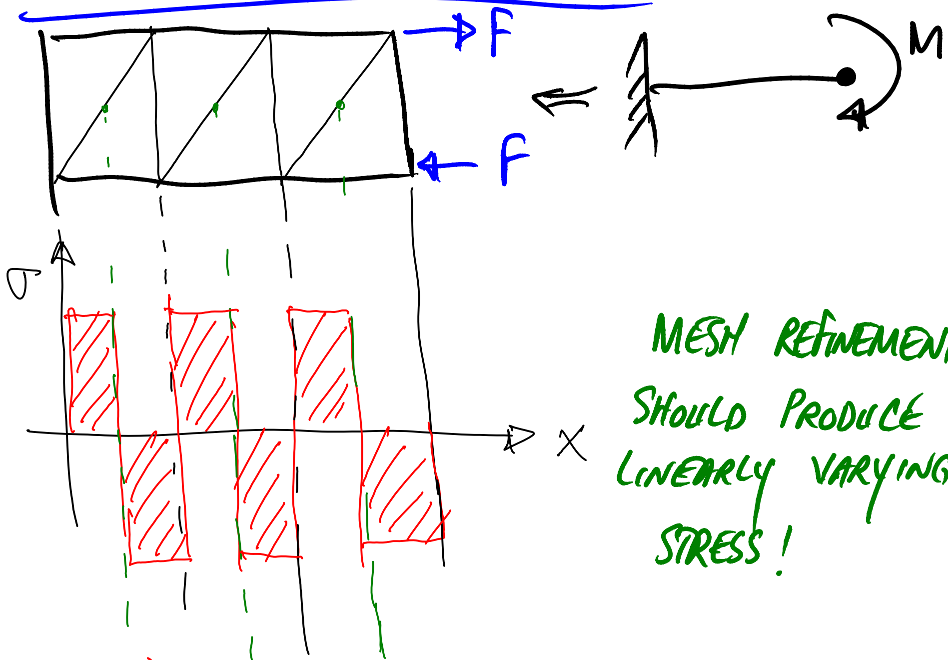
SPECIFY t

MODEL

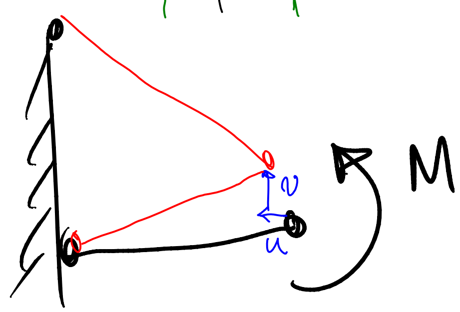


SPECIFY t, E, ν

BEAM UNDER PURE BENDING :



MESH REFINEMENT SHOULD PRODUCE LINEARLY VARYING STRESS !



$\epsilon_x = \text{CONST.}$

$\gamma_{xy} \neq 0$ $\quad a_3 + a_5$

BUT IT SHOULD BE ZERO !

CST WILL ALWAYS PRODUCE

- A SPURIOUS SHEAR TERM (γ_{xy}) WHICH SHOULD NOT BE THERE, UNDER PURE BENDING.

- CST ELEMENTS ARE OVERLY STIFF, AND SUCCESSIVE MESH REFINEMENTS, WHILE CONVERGING, MAY NOT PRODUCE CORRECT RESULTS.

MESH REFINE