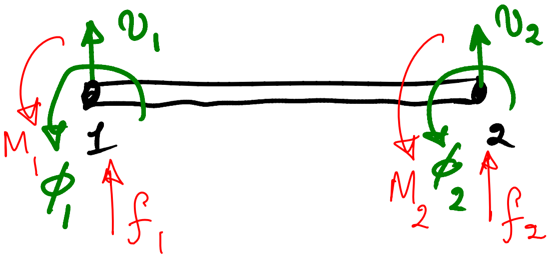
CHAP. 4 (BEAM ELEMENTS)

- BEAM ALLOWS FOR FLEXURE (BENDING).

- SIMPLEST MEMBER OF BEAM HAS 4 D.O.F.'S

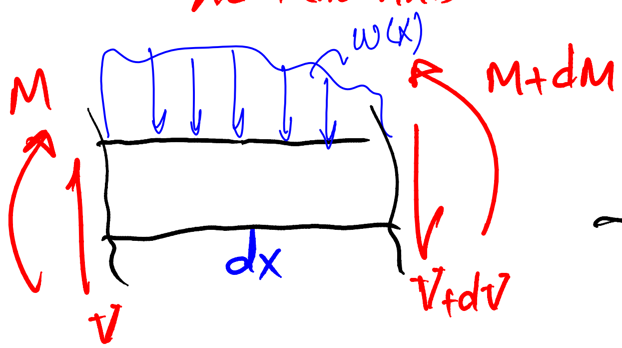


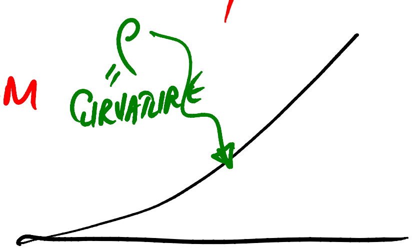
BEAM FORMULATIONS:

1) EULER (BERNOULI) BEAM — LONG, SLENDER
   - TRANSVERSE SHEAR IS IGNORED.
   - PLANE X-SECTIONS REMAIN PLANE AND PERP. TO NEUTRAL AXIS

2) TIMOSHENKO (MINDLIN) BEAM – SHORT, THICK

   TRANSV. SHEAR IS IMPORTANT, PLANE X-SECTIONS REMAIN PLANE, NOT NECESSARILY PERP. TO NEUTRAL AXIS





$$V = \frac{dM}{dx} \qquad \frac{1}{\rho} = \frac{M}{EI} \qquad k = \frac{d^2v}{dx^2}$$

CURVATURE

$$\boxed{\frac{M}{EI} = \frac{d^2v}{dx^2}}$$

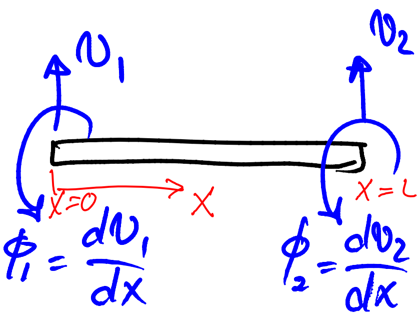
$\Rightarrow$ IN TERMS OF DISTRIBUTED LOAD:

$$\frac{d^2}{}\left(EI\frac{d^2v}{}\right) = -w(x)$$

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2v}{dx^2}\right) = -w(x)$$

ASSUMING E, I ARE CONSTANT FOR EACH ELEM.

$$\boxed{EI\,\frac{d^4v}{dx^4} = -w(x)}$$



ASSUMED DISPLACEMENT:

$$v(x) = a_1 + a_2x + a_3x^2 + a_4x^3$$

MIN. POLYNOMIAL ORDER

APPLYING B.C.'S $\Rightarrow \begin{cases} x=0 & \phi = \frac{dv_1}{dx} \\ v = v_1 \end{cases}$ $\begin{cases} x=L & \phi = \frac{dv_2}{dx} \\ v = v_2 \end{cases}$

$$\frac{dv}{dx} = a_2 + 2a_3x + 3a_4x^2$$

SOLVE TO OBTAIN: $v(x) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix}$

$$v(x) = [N]\{d\}$$

$N_1 = \frac{1}{L^3}(2x^3 - 3x^2L + L^3)$ $\quad N_3 = \cdots$

$\frac{dN_1}{dx} = \frac{1}{L^3}(6x^2 - 6xL)$ $\quad \frac{d^2N_1}{dx^2} = \frac{1}{L^3}(12x - 6L)$

$N_2 = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)$ $\quad N_4 = \cdots$

FORM STRAIN ENERGY: $U = \frac{1}{2}\int_{Vol} \{\sigma\}^T\{\epsilon\}\,dVol.$

$$\{\sigma\} = [E]\{\epsilon\}$$

$$\epsilon_{BEAM} = -y\,\frac{d^2v}{dx^2}$$

$$\{\epsilon\} = [B]\{d\}(-y)$$

STRAIN-DISP. MATRIX

$$v = [N]\{d\} \Rightarrow \frac{dv}{dx} = \frac{d[N]}{dx}\{d\} \Rightarrow \frac{d^2v}{dx^2} = \frac{d^2[N]}{dx^2}\{d\}$$

$$\frac{d^2[N]}{dx^2} = \left[\frac{1}{L^3}(12x - 6L) \quad \cdots \quad \cdots \quad \cdots\right]$$

$$\{\epsilon\} = -y \underbrace{\frac{d^2[N]}{dx^2}}_{[B]}\{d\} \qquad \sigma = E\{\epsilon\} = E\left(-y\frac{d^2[N]}{dx^2}\{d\}\right)$$

$$\{\sigma\}^T = E\{d\}^T[B]^T(-y)\,[B]$$

$$U = \frac{1}{2}\int_{Vol}\left(-yE\{d\}^T[B]^T(-y)[B]\{d\}\right)d\,Vol.$$

$$U = \frac{1}{2}\int_{Vol.}\underbrace{\left(-yE\{d\}^T[B]^T\right)}_{\{\sigma\}^T}\cdot\underbrace{\left(-y[B]\{d\}\right)}_{\{\epsilon\}}\underbrace{dA\,dx}_{dVol.}$$

$$U = \frac{1}{2}\int_A y^2\,dA\cdot\int_{Length}E\{d\}^T[B]^T[B]\{d\}\,dx$$

I = AREA MOMENT OF INERTIA

$$U = \frac{1}{2}\{d\}^T\underbrace{\left(EI\int_0^L[B]^T[B]\,dx\right)}\{d\}$$

COMPARE:

$$U = \frac{1}{2}\{x\}^T[K]\{x\}$$

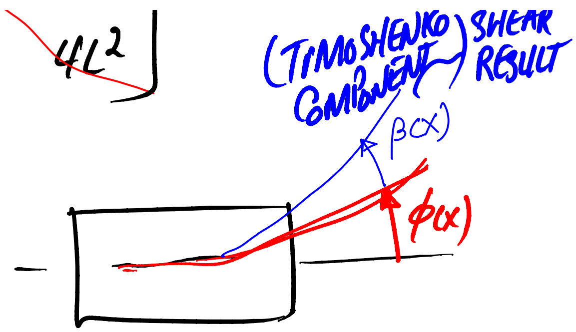
$$[K]_{BEAM} = EI\int_0^L\underset{4\times1}{[B]^T}\,\underset{1\times4}{[B]}\,dx$$

$$[B] = \begin{bmatrix}12x-6L & 6xL-4L^2 & -12x+6L & 6xL-2L^2\end{bmatrix}$$

$$[K] = \frac{EI}{L^3}\begin{array}{c|cccc} & v_1 & \phi_1 & v_2 & \phi_2 \\ \hline v_1 & 12 & 6L & -12 & 6L \\ \phi_1 & 6L & 4L^2 & -6L & 2L^2 \\ v_2 & -12 & -6L & 12 & -6L \end{array}$$

$$\left. \cancel{\tfrac{\varphi}{2}}\, 6L \quad 2L^2 \quad -6L \quad \cancel{4L^2} \right]$$

TIMOSHENKO (MINDLIN)



$$\frac{dv}{dx} = \phi(x) + \beta(x)$$

$$M(x) = EI\,\frac{d\phi(x)}{dx}$$

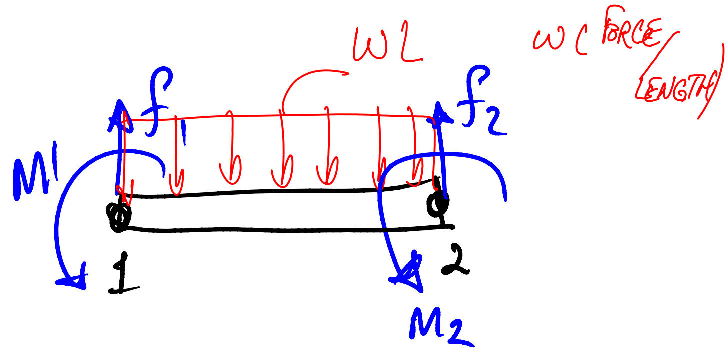
$$V(x) = k_s \cdot A \cdot G \cdot \beta(x)$$

k_s → SHEAR STRESS MULT. FACTOR; A → AREA; G → SHEAR MODULUS; $\beta(x)$ → TIMOSHENKO COMPONENT

$$[k]_{\text{TIMOSHENKO}} = \frac{EI}{L^3(1+\varphi)} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\varphi)L^2 & -6L & (2-\varphi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\varphi)L^2 & -6L & (4+\varphi)L^2 \end{bmatrix}$$

$$\varphi = \frac{12EI}{k_s A G L^2}$$

DISTRIBUTED LOAD



ALL FORCE TERMS:

$$\{\hat{f}\} = -\int_{Vol.} \hat{X}\,[N]\,\{\hat{d}\}\,dVol - \int_{SUR.} \hat{T}\,[N]\,\{\hat{d}\}\,dA - \{\hat{P}\}$$

$\hat{X}$ → Vol. FORCE; $\hat{T}$ → SURF. TRACTION; $\hat{P}$ → POINT LOAD

$$\text{DIST. LOAD} = \int_0^L w(x)\,[N]^T\,dx$$

WORK-EQUIVALENT METHOD:

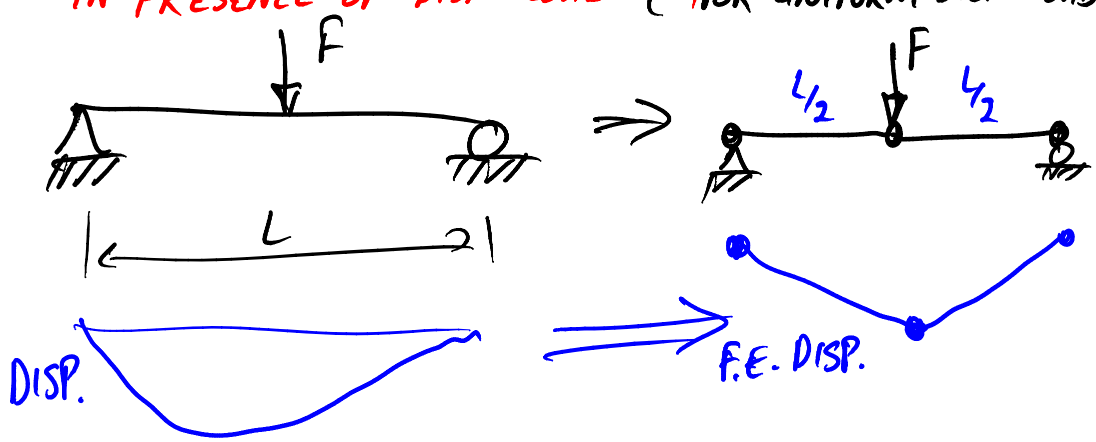
WORK OF DISTRIBUTED LOAD IS EQUIV. TO WORK DONE BY FORCES AND MOMENTS APPROXIMATED AS F.E. LOADS.

$$W_{F.E} = m_1 \hat{\phi}_1 + m_2 \hat{\phi}_2 + f_1 \hat{d}_1 + f_2 \hat{d}_2$$

$$W_{\substack{DIST.\\LOAD}} = \int_0^L w(x)\, v(x)\, dx$$

POLYNOMIAL ORDER FOR DISTRIBUTED LOAD IS, AT LEAST ONE ORDER HIGHER THAN DISP. FIELD.

THEREFORE, USE TWICE AS MANY ELEM. DISTRIBUTIONS IN PRESENCE OF DIST. LOAD. (FOR UNIFORM DIST. LOAD).



NOTE: F.E. MODEL DOES NOT REPRESENT DEFORMED SHAPE WITH MINIMUM ELEMENT DISCRETIZATION.

USE 3 ELEMENTS PER SPAN TO OBTAIN "SMOTH" DISP. FIELD.