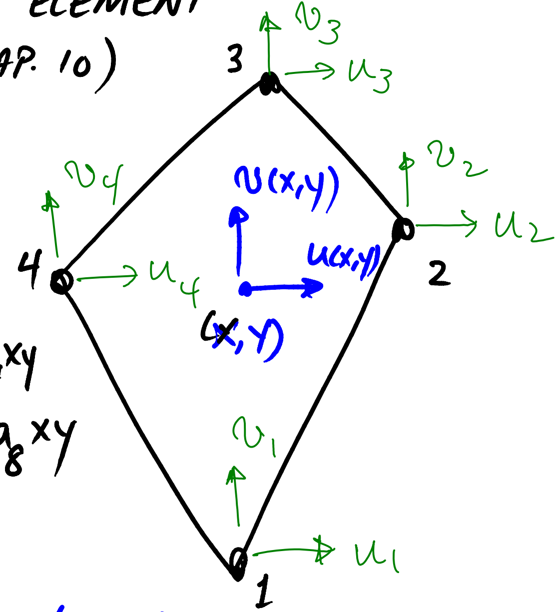
4-NODE QUADRILATERAL ELEMENT
(VARIATION ON CHAP. 10)

8 TOTAL D.O.F.'S



SIMPLEST POLYNOMIAL DISP. APPROXIMATION

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$
$$v(x,y) = a_5 + a_6 x + a_7 y + a_8 xy$$

$$\epsilon_x = \frac{\partial u}{\partial x} = a_2 + a_4 y$$
$$\epsilon_y = \frac{\partial v}{\partial y} = a_7 + a_8 x$$

BI-LINEAR ELEMENT

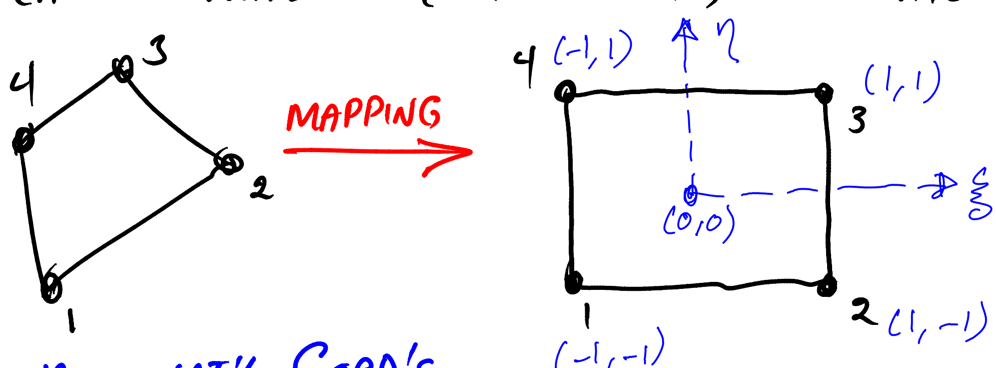
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + a_4 x + a_6 + a_8 y$$

NO LONGER CONST.

ISOPARAMETRIC ELEMENT:

1) WE USE SAME SHAPE FUNCTIONS (EMPLOYED FOR DISP. APROX.) TO DESCRIBE X- AND Y-COORDINATES OF POINT OF INTEREST WITHIN ELEMENT.

2) MAP THE NODAL COORD'S TO A NEW SET OF COORD'S CALLED NATURAL (OR INTRINSIC) COORDINATES.



$\xi - \eta \equiv$ NAT'L COORD'S

SHAPE FUNCTIONS (BASED ON NAT'L COORD. SYSTEM):

$N_1 = C(1-\xi)(1-\eta)$ C's ARE DETERMINED BY

$N_1 = C_1(1-\xi)(1-\eta)$

$N_2 = C_2(1+\xi)(1-\eta)$

$N_3 = C_3(1+\xi)(1+\eta)$

$N_4 = C_4(1-\xi)(1+\eta)$

C_i's ARE DETERMINED BY APPLYING B.C.'S:

e.g., $N_1 = 1$ WHEN $\begin{cases} \xi = -1 \\ \eta = -1 \end{cases}$ @ NODE 1

$$\boxed{C_1 = \frac{1}{4}}$$

$$\{u\} = [N]\{d\} \Rightarrow \underset{2\times 1}{\begin{Bmatrix} u \\ v \end{Bmatrix}} = \underset{2\times 8}{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}} \underset{8\times 1}{\begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}}$$

ISOPARAMETRIC CONCEPT:

COORD'S OF POINT OF INTEREST:

$$X = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$Y = N_1 Y_1 + N_2 Y_2 + N_3 Y_3 + N_4 Y_4$$

$$f(x,y) = f\left[x(\xi,\eta),\ y(\xi,\eta)\right]$$

$$\begin{cases} \dfrac{\partial f}{\partial \xi} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial \xi} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial f}{\partial \eta} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial \eta} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial \eta} \end{cases}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\[1ex] \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\[1ex] \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = -(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\[1ex] \dfrac{\partial f}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\[1ex] \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [B] \{d\}$$

3X1 $\quad$ $\boxed{3X8}$ $\quad$ 8X1

$$[B] = [A][G]$$

3X4

$$A = \frac{1}{\det[J]} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$[G] = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \end{bmatrix}$$

4X8

$$\{\epsilon\} = [B]\{d\}$$

$$\{\sigma\} = [E][B]\{d\} \Rightarrow \{\sigma\}^T = \{d\}^T [B]^T [E]^T$$

$$U = \frac{1}{2} \int_{vol.} \{\sigma\}^T \{\epsilon\} \, d\,vol. \qquad d\,vol. = \det[J] \, d\xi \, d\eta$$

$$U = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \{d\}^T [B]^T [E]^T [B] \{d\} \det[J] \, d\xi \, d\eta$$

$$U = \frac{1}{2} \{d\}^T \left(\int_{-1}^{1} \int_{-1}^{1} [B]^T [E]^T [B] \det[J] \, d\xi \, d\eta \right) \{d\}$$

COMPARE

$$\frac{1}{2} \{x\}^T \cdot [K] \cdot \{x\}$$

$$[k] = \int^{1} \int^{1} [B]^T [E]^T [B] \, d\xi \, d\eta$$

INTEGRATION

NOT TRIVIAL

NUMERICAL INTEGRATION