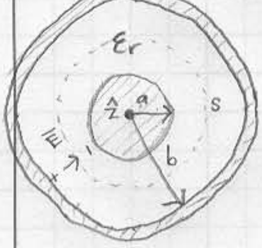
Problem 6-15 Ulaby

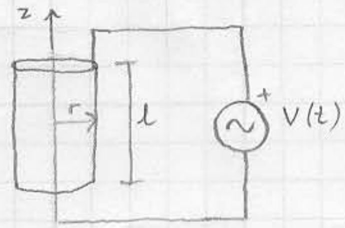
A coaxial capacitor of length $l = 0.06$ m uses an insulating dielectric material with $\varepsilon_r = 9$. The radii of the cylindrical conductors are 0.005 m and 0.01 m.

If the voltage applied across the capacitor is $V(t) = 50 \sin(120\pi t)$ V, what is the displacement current?



cross-section of coaxial capacitor

▨ conductors
☐ dielectric



side-view of coaxial capacitor

$$\nabla \times \bar{H} = J + \frac{\partial \bar{D}}{\partial t}$$ Amperes Law

$$\int_s (\nabla \times \bar{H}) \cdot d\bar{s} = \int_s J \cdot d\bar{s} + \int_s \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$$

$$\oint_c \bar{H} \cdot d\bar{l} = I_c + \int_s \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$$

stokes theorem

$$I_d \triangleq \int_s J_d \cdot d\bar{s} = \int_s \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s} \qquad (6.44)$$

using Gauss's law to determine the electric field through a closed surface inside the dielectric.

$$\oint_s \bar{D} \cdot d\bar{s} = Q \qquad \text{where } Q = \rho_l l$$

$$\int_0^l \int_0^{2\pi} -\hat{r} D_r \cdot \hat{r}\, r\, d\phi\, dz = \int_0^l dz \int_0^{2\pi} d\phi\, -D_r r = l\, 2\pi(-D_r) r$$

$$-D_r = \frac{Q}{2\pi l r} \Rightarrow \bar{E} = \frac{\bar{D}}{\varepsilon} = -\hat{r}\frac{D_r}{\varepsilon} = -\hat{r}\frac{Q}{2\pi l r \varepsilon} \qquad \text{(pg. 185 ulaby)}$$

The potential difference between the inner and outer conductors,

$$V_c = -\int_a^b \bar{E} \cdot d\bar{l} = -\int_a^b \left(-\hat{r}\frac{Q}{2\pi l r \varepsilon}\right) \cdot d\bar{l} = \int_a^b \hat{r}\frac{Q}{2\pi l r \varepsilon} \cdot \hat{r}\, dr = \frac{Q}{2\pi l \varepsilon}\left(\ln(r)\right)\Big|_a^b$$

$$= \frac{Q}{2\pi l \varepsilon} \ln(b) - \ln(a) = \frac{Q}{2\pi l \varepsilon} \ln\left(\frac{b}{a}\right)$$

Note: $V = Ed$ (4.117 ulaby)

$$d = \frac{V}{E}$$

$$d = \frac{V_c}{E_r} = \frac{Q \ln(b/a)}{2\pi l \varepsilon} \cdot \frac{2\pi l r \varepsilon}{Q} = r \ln(b/a)$$

$$\bar{E} = -\hat{r}\frac{V}{d} = -\hat{r}\frac{V_0 \sin(wt)}{r \ln(b/a)}$$

Now we can determine the displacement current using $\varepsilon \bar{E} = \bar{D}$,

$$I_d = \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s} = \int_S \frac{\partial}{\partial t}\left(\varepsilon \bar{E}\right) \cdot d\bar{s} = \int_0^{l} \int_0^{2\pi} \frac{\partial}{\partial t}\left(\varepsilon \left(-\hat{r} \frac{V_0 \sin(wt)}{r \ln(b/a)}\right)\right) \cdot \hat{r}\, r\, d\phi\, dz$$

$$= \int_0^{l} dz \int_0^{2\pi} d\phi \left(-\frac{\varepsilon V_0}{\ln(b/a)} \frac{\partial}{\partial t} \sin(wt)\right)$$

$$\boxed{= -l\, 2\pi \frac{\varepsilon V_0}{\ln(b/a)} w \cos(wt)}$$

$$I_d = -\frac{2\pi \cdot 0.06 \cdot 9 \cdot 8.85 \times 10^{-12} \cdot 50 \cdot 120\pi}{\ln(0.5/1)} \cos(120\pi t)$$

$$= 816.57 \times 10^{-9} \cos(120\pi t)$$

$$= 0.816 \cos(120\pi t) \quad \mu A$$