Section 4.1: Properties of Binary Relations

A "binary relation" R over some set A is a subset of A×A.
If (x,y) ∈ R we sometimes write x R y.

Example: Let R be the binary relaion "less" ("<") over $\mathbb{N}$.

{(0,1), (0,2), … (1,2), (1,3), … }
(4,7) ∈ R
Normally, we write: 4 < 7

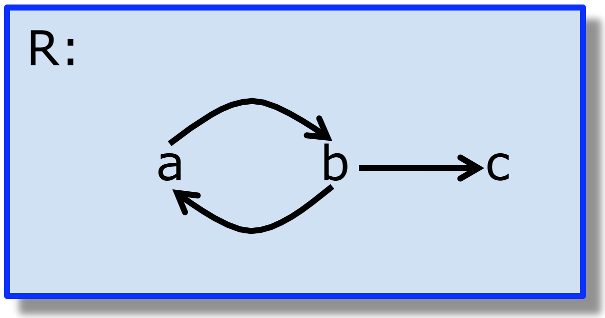
Additional Examples: Here are some binary relations over A={0,1,2}

Ø *(nothing is related to anything)*
A×A *(everything is related to everything)*
eq = {(0,0), (1,1),(2,2)}
less = {(0,1),(0,2),(1,2)}

Representing Relations with Digraphs (directed graphs)

Let R = {(a,b), (b,a), (b,c)} over A={a,b,c}

We can represent R with this graph:



Properties of Binary Relations:

R is reflexive

x R x for all $x \in A$

Every element is related to itself.

R is symmetric

x R y implies y R x, for all $x,y \in A$

The relation is reversable.

R is transitive

x R y and y R z implies x R z, for all $x,y,z \in A$

Example:

$i<7$ and $7<j$ implies $i<j$.

R is irreflexive

$(x,x) \notin R$, for all $x \in A$

Elements aren't related to themselves.

R is antisymmetric

x R y and y R x implies that $x=y$, for all $x,y,z \in A$

Example: $i \leq 7$ and $7 \leq i$ implies $i=7$.

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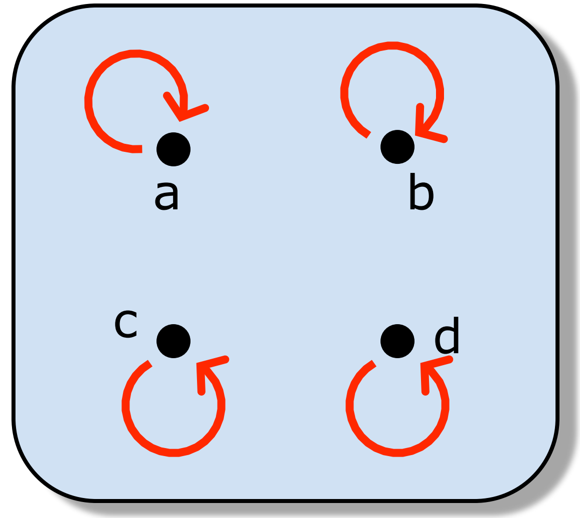

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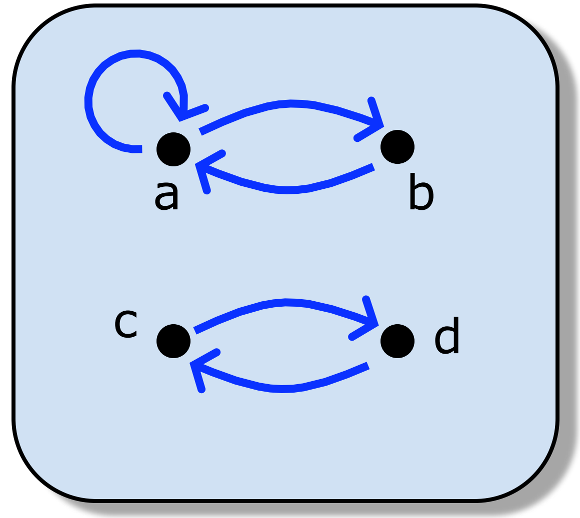

Symmetric:

All edges are 2-way:
Might as well use
undirected edges!

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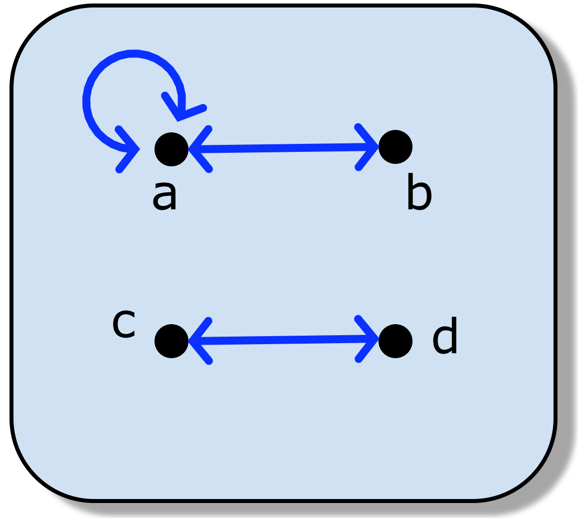

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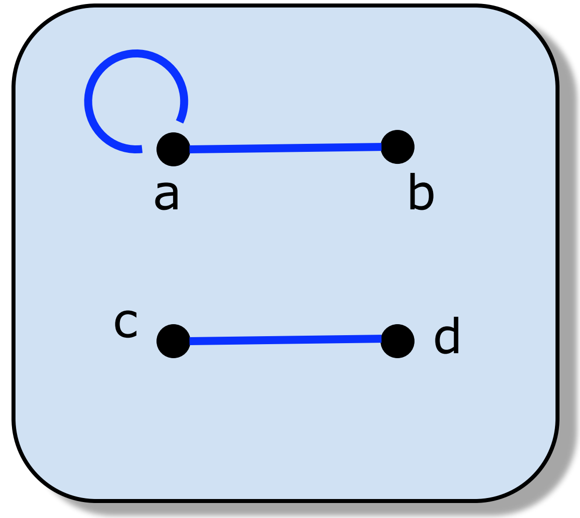

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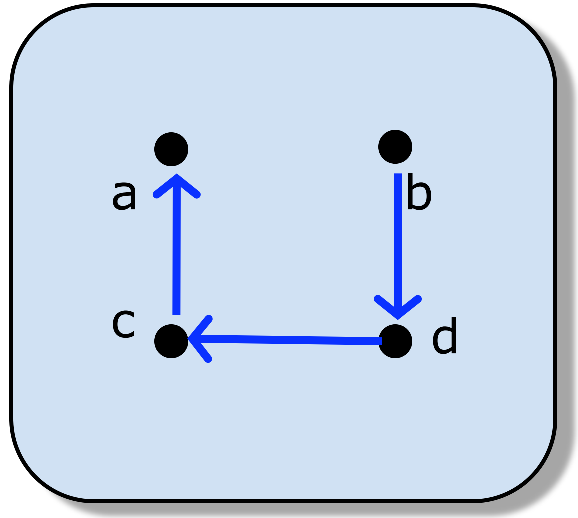

Transitive:
If you can get from x to y, then there is an edge directly from x to y!

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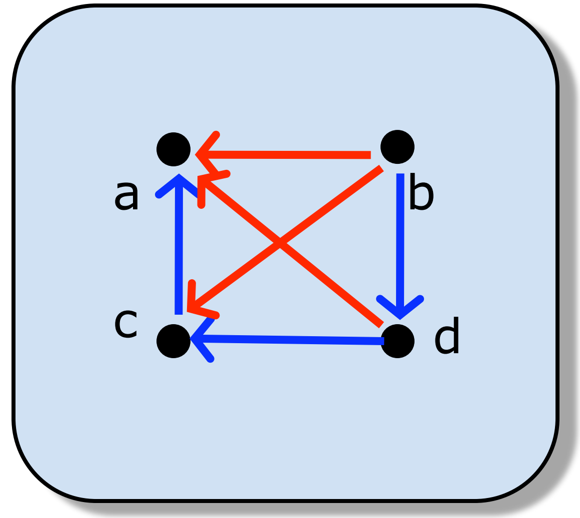

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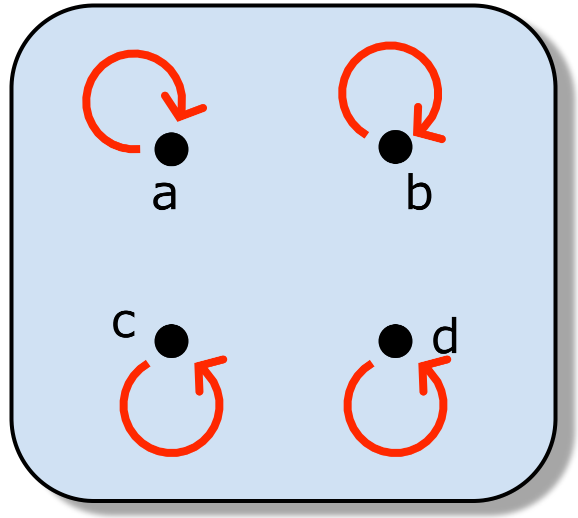

Irreflexive:

You won't see any edges like these!

Properties of Binary Relations:

R is reflexive

$x \; R \; x$ for all $x \in A$

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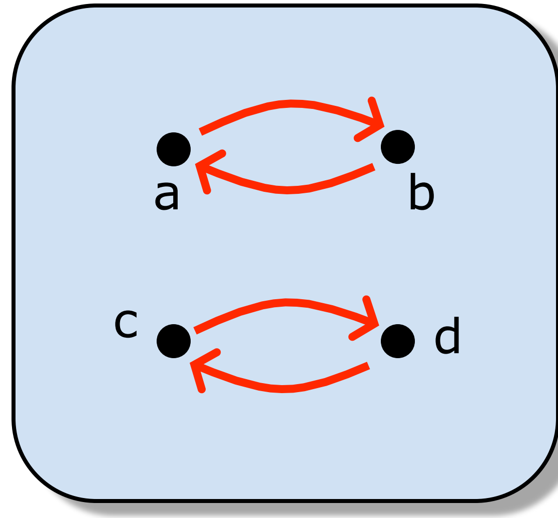
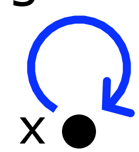


Antisymmetric:

You won't see any edges like these! (although xRx is okay:



)

Properties of Binary Relations:

R is reflexive	x R x for all $x \in A$
R is symmetric	x R y implies y R x, for all $x,y \in A$
R is transitive	x R y and y R z implies x R z, for all $x,y,z \in A$
R is irreflexive	$(x,x) \notin R$, for all $x \in A$
R is antisymmetric	x R y and y R x implies that x=y, for all $x,y,z \in A$

Examples: Here are some binary relations over A={0,1}.
Which of the properties hold?

Answers:

Ø
A×A
eq = {(0,0), (1,1)}
less = {(0,1)}

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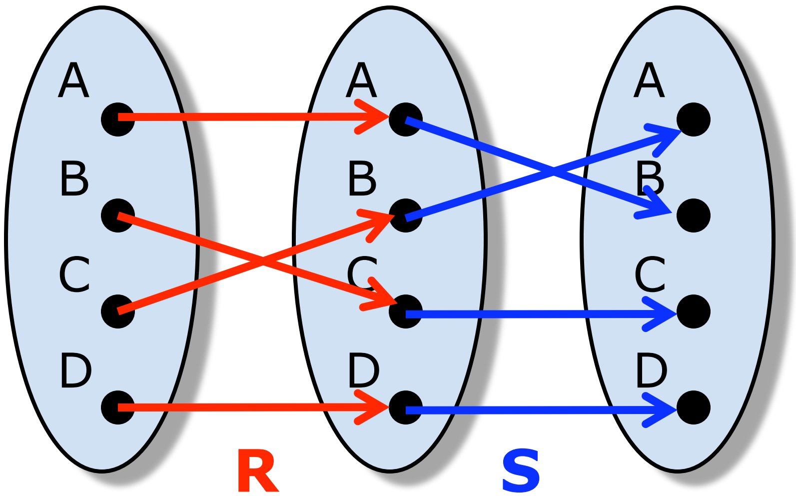
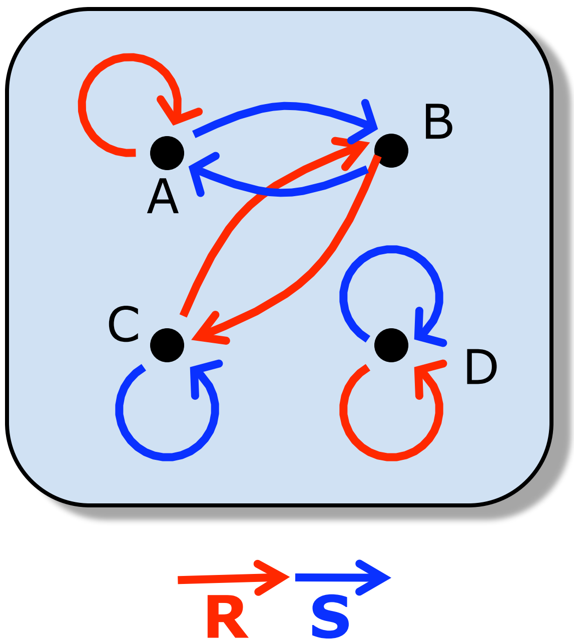
Examples: Here are some binary relations over A={0,1}.
Which of the properties hold?

	Answers:
∅	symmetric,transitive,irreflexive,antisymmetric
$A \times A$	reflexive, symmetric, transitive
eq = {(0,0), (1,1)}	reflexive, symmetric, transitive, antisymmetric
less = {(0,1)}	transitive, irreflexive, antisymmetric

Composition of Relations

If R and S are binary relations, then the composition of R and S is
$R \circ S = \{(x,z) \mid x \; R \; y \text{ and } y \; S \; z \text{ for some } y\}$





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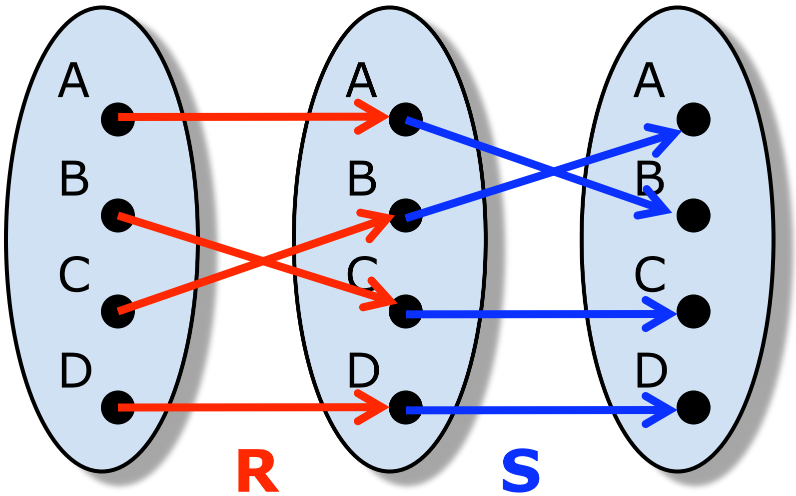

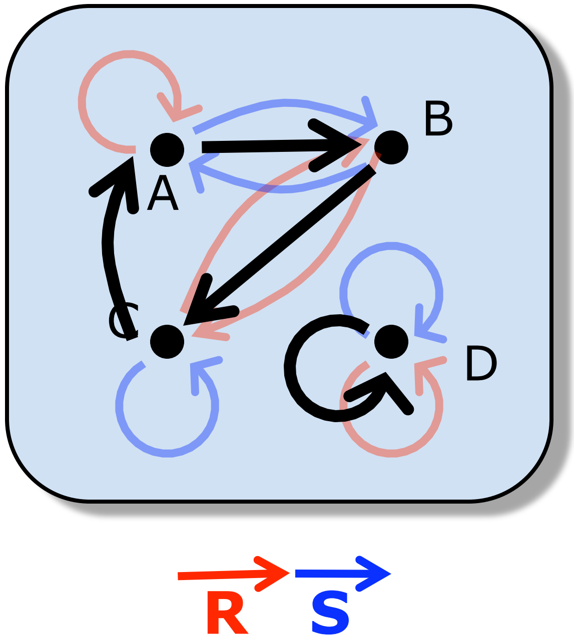

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eq ∘ less = ?

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isMotherOf ∘ isFatherOf = ?

isSonOf ∘ isSiblingOf = ?

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eq ∘ less = less

{ (x,z) | x=y and y<x, for some y}

R ∘ Ø = Ø

isMotherOf ∘ isFatherOf = isPaternalGrandmotherOf

{ (x,z) | x isMotherOf y and y isFatherOf x, for some y}

isSonOf ∘ isSiblingOf = isNephewOf

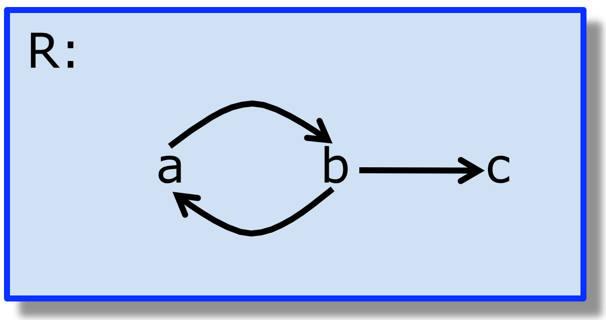
{ (x,z) | x isSonOf y and y isSiblingOf x, for some y}

Representing Relations with Digraphs (directed graphs)

Let R = {(a,b), (b,a),(b,c)} over A={a,b,c}

Let $R^2 = R \circ R$ = ?

We can represent R graphically:



Representing Relations with Digraphs (directed graphs)

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Let $R^3 = R \circ R \circ R$ = ?

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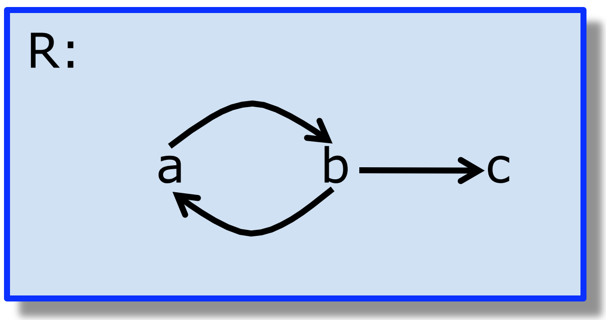

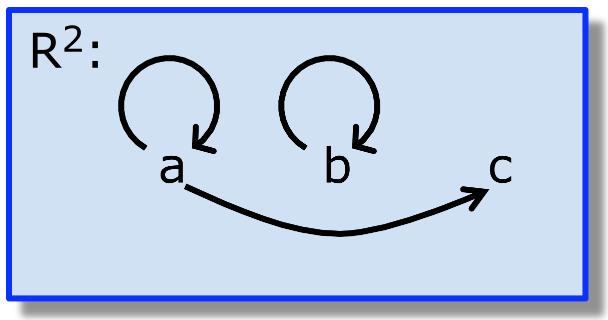

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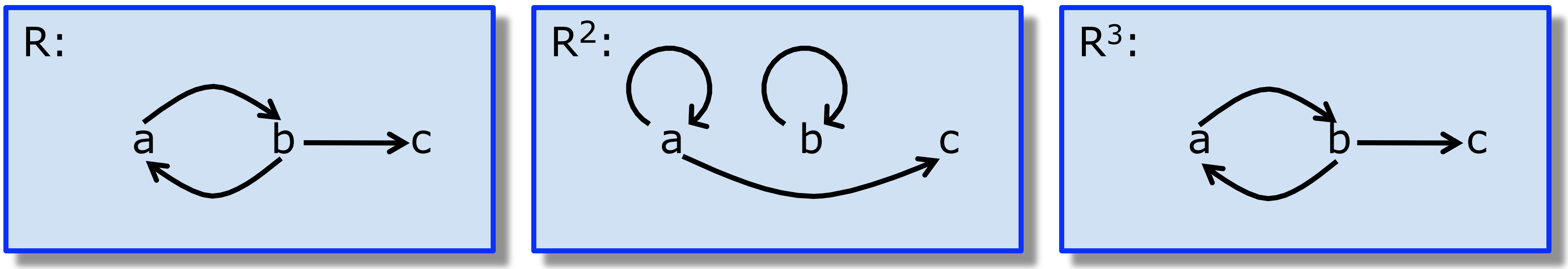


In this example, R^3 happens to be the same relation as R.
$R^3 = R$

Note: By definition, $R^0 = Eq$, where x Eq y iff x=y.

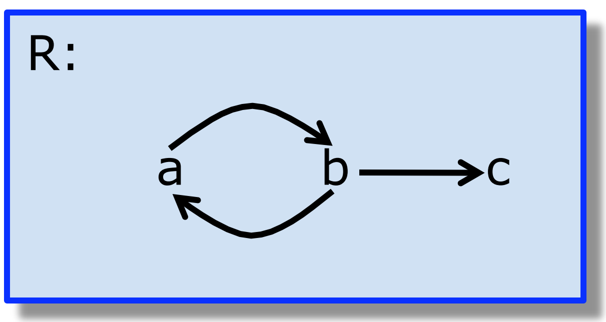
Reflexive Closure

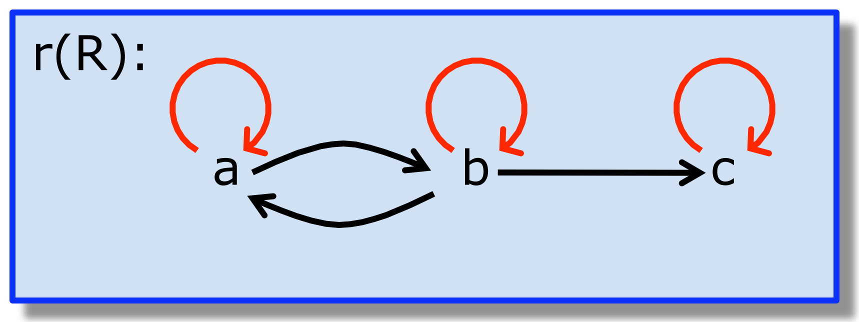
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the reflexive property.

Reflexive Closure of R is r(R) = R ∪ Eq, where Eq is the equality relation.

Example:

r(R) = R ∪ Eq = {(a,b),(b,a),(b,c),(a,a),(b,b),(c,c)}





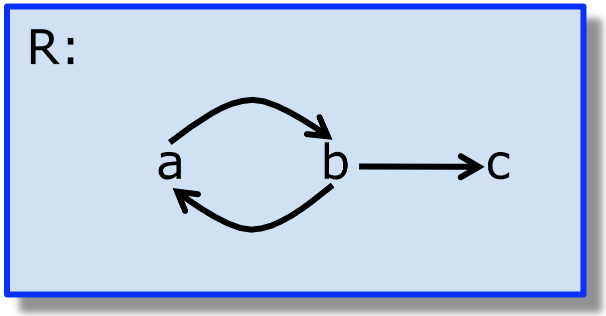
Symmetric Closure

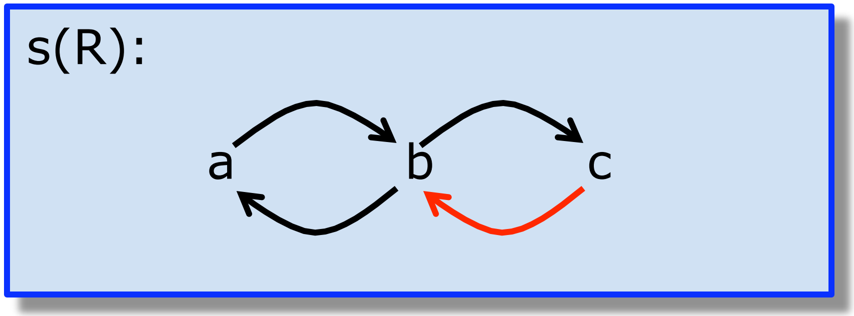
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the symmetric property.

Symmetric Closure of R is s(R) = $R \cup R^c$, where R^c is the converse relation. $R^c = \{(b,a) \mid a \; R \; b\}$

Example:

$s(R) = R \cup R^c = \{(a,b),(b,a),(b,c),(c,b)\}$





Transitive Closure

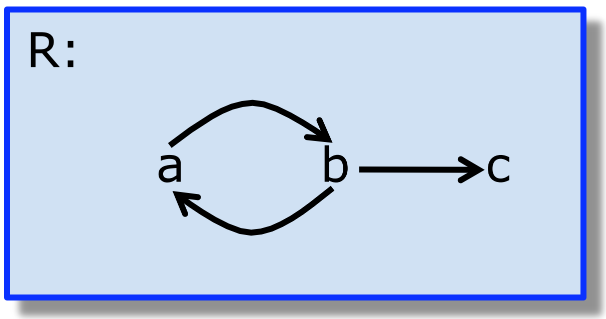
Given a relation R, we want to add to it just enough "edges" to make the resulting relation satisfy the transitivity property.

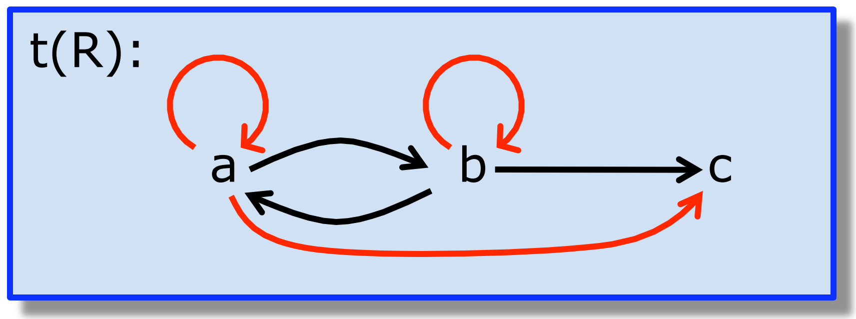
Transitive Closure of R is $t(R) = R \cup R^2 \cup R^3 \cup \ldots$

Note: If the number of nodes is finite…
If $|A| = n$ then $t(R) = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$

Example:

$t(R) = R \cup R^2 \cup R^3 = \{(a,b),(b,a),(b,c),(a,a),(b,b),(a,c)\}$





If there is a path from x to y, then add an edge directly from x to y.

In-Class Quiz:

Let R = {(x,x+1) | x$\in\mathbb{Z}$ }

What is t(R)?
What is rt(R)?
What is st(R)?

In-Class Quiz:

Let $R = \{(x,x+1) \mid x \in \mathbb{Z}\}$

What is t(R)? $<$
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What is rt(R)? $\leq$
What is st(R)? $\neq$

Adjacency Matrix

Idea: Use a matrix to represent a directed graph (or a relation).

Let R = {(a,b),(b,c),(c,d)}

Number the elements in the set: 1,2,3,…

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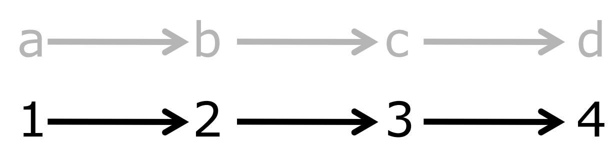

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Number the elements in the set: 1,2,3,…

Now we can write R = {(1,2),(2,3),(3,4)}

The matrix M will have a "1" in position M_{ij} if i R j, and "0" otherwise.

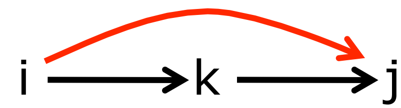
$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Computing the Transitive Closure: Warshall's Algorithm

We can use this matrix to compute the t(R), the transitive closure of R.

Idea: Every time we find this pattern:

…add this edge:



Warshall's Algorithm

```
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
            if M_ik = M_kj = 1 then
                M_ij := 1
            endIf
        endFor
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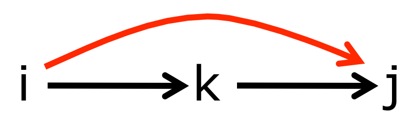
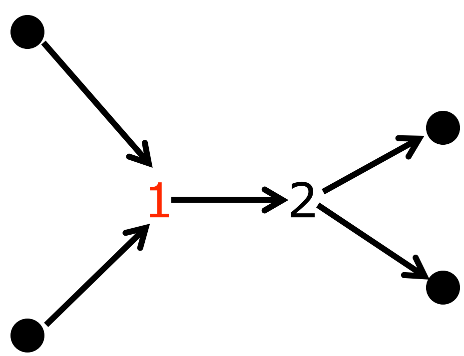


…add this edge:

Consider all ways to bypass node 1.
Then forget about node 1.
Consider all ways to bypass node 2.
and so on…



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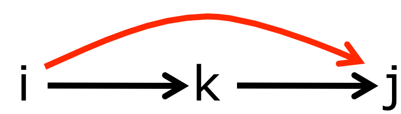

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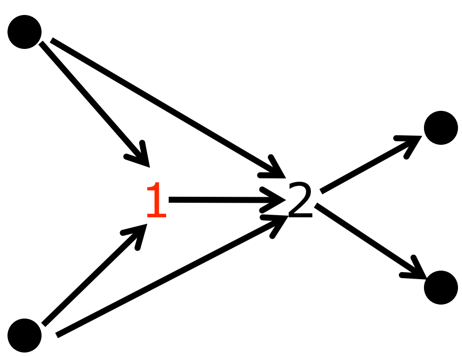

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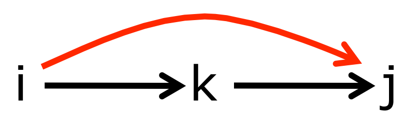

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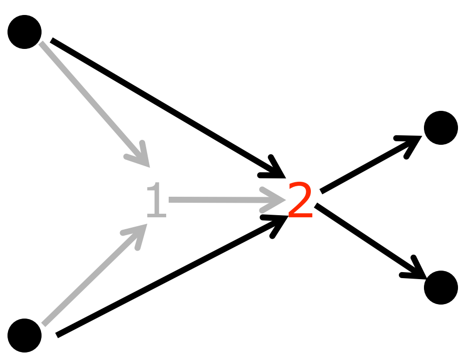
Consider all ways to bypass node 2.
and so on…



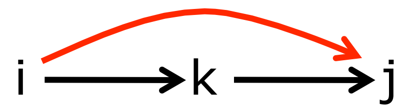
Warshall's Algorithm

for k := 1 **to** n
    for i := 1 **to** n
        for j := 1 **to** n
            if $M_{ik} = M_{kj} = 1$ **then**
                $M_{ij} := 1$
            endIf
        endFor
    endFor
endFor

Computing the Transitive Closure: Warshall's Algorithm

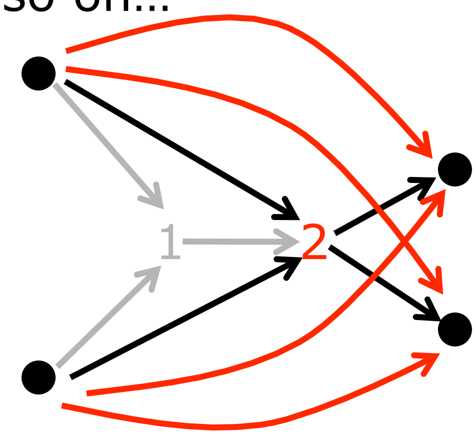
We can use this matrix to compute the t(R), the transitive closure of R.

Idea: Every time we find this pattern:



…add this edge:

Consider all ways to bypass node 1.
Then forget about node 1.
Consider all ways to bypass node 2.
and so on…



Warshall's Algorithm

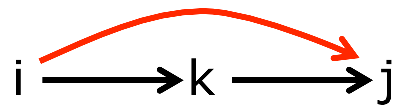
```
for k := 1 to n
   for i := 1 to n
      for j := 1 to n
         if M_ik = M_kj = 1 then
               M_ij := 1
         endIf
      endFor
   endFor
endFor
```

Computing the Transitive Closure: Warshall's Algorithm

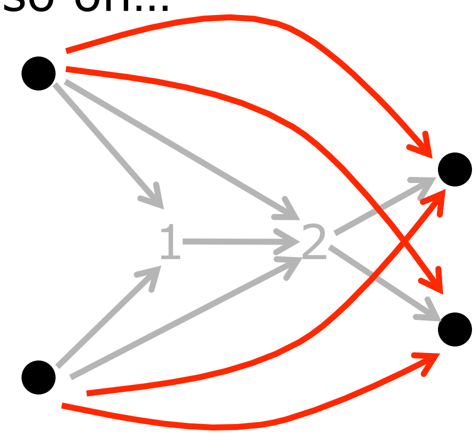
We can use this matrix to compute the t(R), the transitive closure of R.

Idea: Every time we find this pattern:

…add this edge:



Consider all ways to bypass node 1.
Then forget about node 1.
Consider all ways to bypass node 2.
and so on…



Warshall's Algorithm

```
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
            if M_ik = M_kj = 1 then
                  M_ij := 1
            endIf
        endFor
    endFor
endFor
```

Example:

k=1

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$$

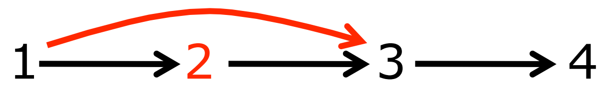
$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

```
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
            if M_ik = M_kj = 1 then
                  M_ij := 1
            endIf
        endFor
    endFor
endFor
```

Example:

k=2



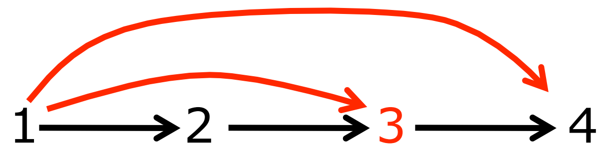
$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

```
for k := 1 to n
    for i := 1 to n
        for j := 1 to n
            if M_ik = M_kj = 1 then
                   M_ij := 1
            endIf
        endFor
    endFor
endFor
```

Example:

k=3



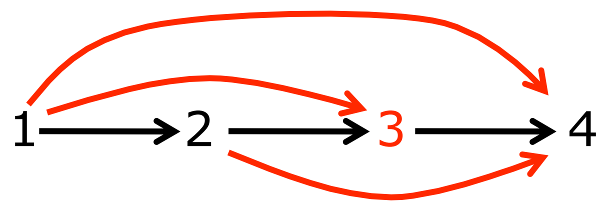
$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

for k := 1 **to** n
    for i := 1 **to** n
        for j := 1 **to** n
            if $M_{ik} = M_{kj} = 1$ **then**
                $M_{ij} := 1$
            endIf
        endFor
    endFor
endFor

Example:

k=3



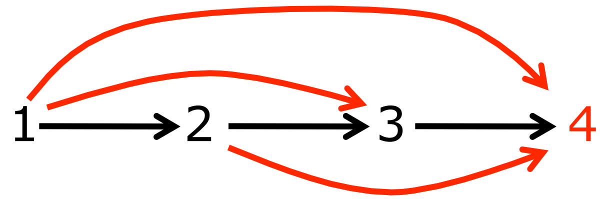
$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

for k := 1 **to** n
    for i := 1 **to** n
        for j := 1 **to** n
            if $M_{ik} = M_{kj} = 1$ **then**
                $M_{ij} := 1$
            endIf
        endFor
    endFor
endFor

Example:

k=4 …Done;
no more changes.



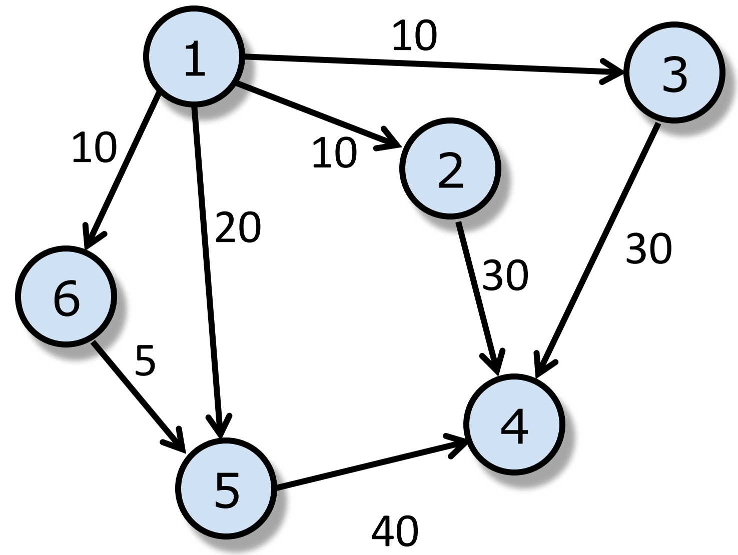
$$M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

for k := 1 **to** n
  for i := 1 **to** n
    for j := 1 **to** n
      if $M_{ik} = M_{kj} = 1$ **then**
        $M_{ij} := 1$
      endIf
    endFor
  endFor
endFor

Path Problems: Floyd's Algorithm

Consider a directed graph with weights on the edges.



Problems:

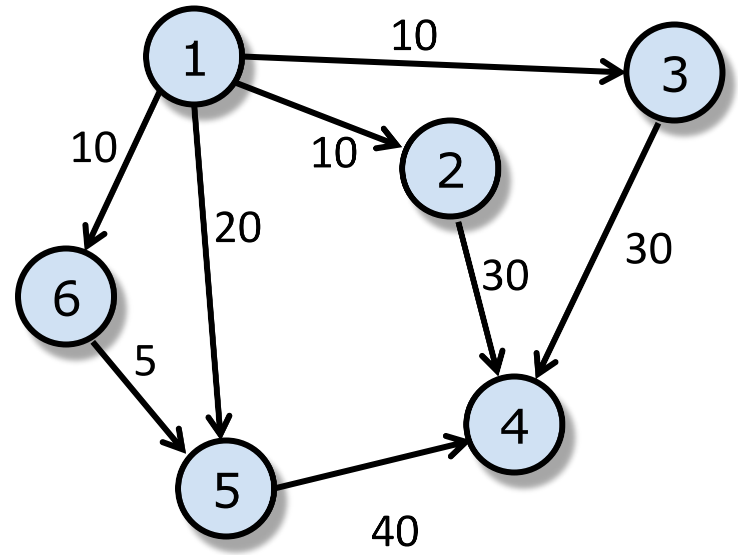
Find the cheapest path from x to y.

Find the shortest path from x to y.

(Just make all weights = 1!)

Path Problems: Floyd’s Algorithm

Consider a directed graph with weights on the edges.



Idea:

Represent the graph with a matrix.
Store the weights.
For non-existent edges, use
a weight of ∞.
Then modify Warshall’s Algorithm.

$$M = \begin{bmatrix} 0 & 10 & 10 & \infty & 20 & 10 \\ \infty & 0 & \infty & 30 & \infty & \infty \\ \infty & \infty & 0 & 30 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 40 & 0 & \infty \\ \infty & \infty & \infty & \infty & 5 & 0 \end{bmatrix}$$

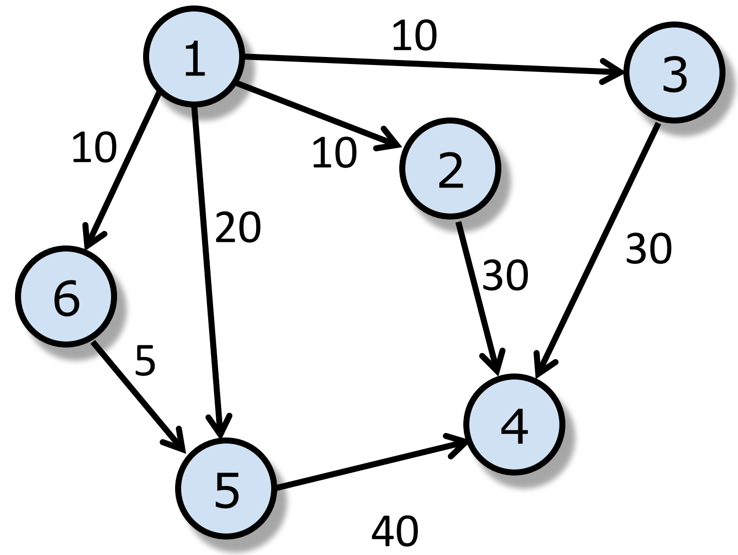
Path Problems: Floyd's Algorithm

Consider a directed graph with weights on the edges.

Floyd's Algorithm:

for k := 1 **to** n
  for i := 1 **to** n
    for j := 1 **to** n
      if $M_{ik}+M_{kj} < M_{ij}$
        $M_{ij} := M_{ik}+M_{kj}$

      endIf
    endFor
  endFor
endFor



$$M = \begin{bmatrix} 0 & 10 & 10 & \infty & 20 & 10 \\ \infty & 0 & \infty & 30 & \infty & \infty \\ \infty & \infty & 0 & 30 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 40 & 0 & \infty \\ \infty & \infty & \infty & \infty & 5 & 0 \end{bmatrix}$$

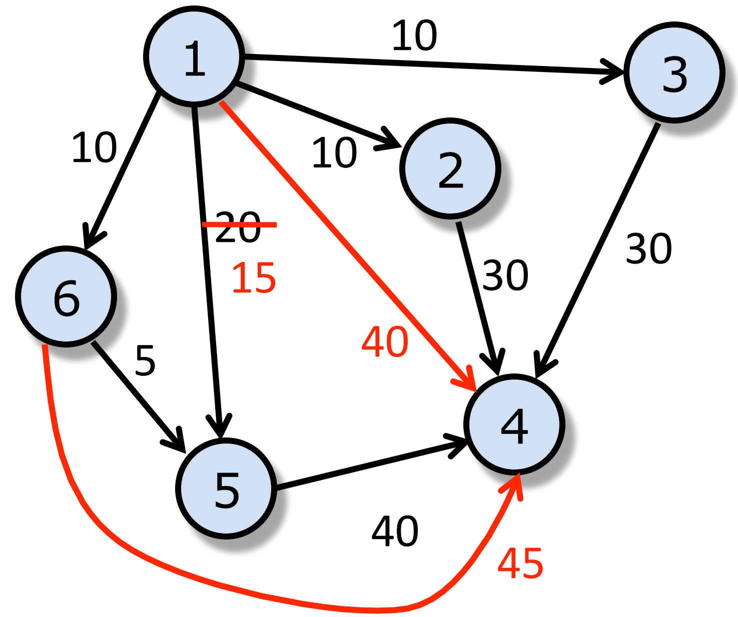
Path Problems: Floyd's Algorithm

Consider a directed graph with weights on the edges.

Floyd's Algorithm:

for k := 1 **to** n
  for i := 1 **to** n
    for j := 1 **to** n
      if $M_{ik}+M_{kj} < M_{ij}$
        $M_{ij} := M_{ik}+M_{kj}$

      endIf
    endFor
  endFor
endFor



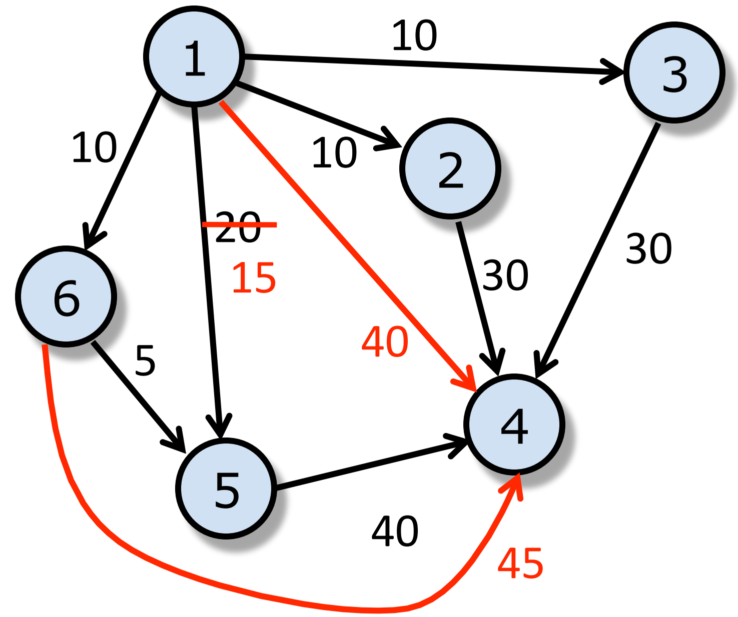
$$M = \begin{bmatrix} 0 & 10 & 10 & 40 & 15 & 10 \\ \infty & 0 & \infty & 30 & \infty & \infty \\ \infty & \infty & 0 & 30 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 40 & 0 & \infty \\ \infty & \infty & \infty & 45 & 5 & 0 \end{bmatrix}$$

Path Problems: Floyd's Algorithm

How can we remember the best path? P=next node in best path!

Floyd's Algorithm:

for k := 1 **to** n
  for i := 1 **to** n
    for j := 1 **to** n
      if $M_{ik}+M_{kj} < M_{ij}$
        $M_{ij} := M_{ik}+M_{kj}$
        $P_{ij} := k$
      endIf
    endFor
  endFor
endFor



$$M = \begin{bmatrix} 0 & 10 & 10 & 40 & 15 & 10 \\ \infty & 0 & \infty & 30 & \infty & \infty \\ \infty & \infty & 0 & 30 & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 40 & 0 & \infty \\ \infty & \infty & \infty & 45 & 5 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \end{bmatrix}$$