Section 1.3 Ordered Structures

Tuples
Have order and can have repetitions.
(6,7,6) is a 3-tuple
() is the empty tuple
A 2-tuple is called a “pair” and a 3-tuple is called a “triple”.
We write \((x_1, \ldots, x_n) = (y_1, \ldots, y_n)\) to mean \(x_i = y_i\) for \(1 \leq i \leq n\).

Cartesian Product:
\(A \times B = \{ (x,y) \mid x \in A \text{ and } y \in B \}\)
This definition extends naturally:
\(A \times B \times C = \{ (x,y,z) \mid x \in A \text{ and } y \in B \text{ and } z \in C \}\)

Notation:
\(A^0 = \{ () \}\)
\(A^1 = \{ (x) \mid x \in A \}\)
\(A^2 = \{ (x_1,x_2) \mid x_2 \in A \text{ and } x_2 \in A \}\)
\(A^n = \{ (x_1, \ldots, x_n) \mid x_i \in A \}\)

In-Class Quiz:
Does \((A \times B) \times C = A \times (B \times C)\)?
Lists
Like tuples but there is no random access.
Example:
   <a,b,c,b> is a list with 4 elements
   <> is the empty list.

List operations: head, tail, cons
   head ( <a,b,c,b> )  =  a
   tail ( <a,b,c,b> )  =  <b,c,b>
   cons ( e, <a,b,c,b> )  =  <e,a,b,c,b>

The set of lists whose elements are in A is denoted by lists(A).

Lists can contain lists:
   < 3 , <a,b,c> , 4 , <7,8> , e , <> , g >

In-class Quiz:
For L = <<<a>,b,<c,d>>>
   Find head(L)
   Find tail(L)
Strings

Like lists.
   All elements come from an alphabet.
   The elements are juxtaposed.
Example: alphabet is $\mathcal{A} = \{a, b\}$.
   Some strings: $a, b, aa, ab, ba, bb, aaa, bbb, ...$

The empty string is denoted by $\Lambda$ (lambda).

**Concatenation** of two strings is their juxtaposition.
   The concatenation of $ab$ and $bab$ is $abbab$.

This is true of any string $s$:
   $s \Lambda = \Lambda s = s$

If $s$ is a string, $s^n$ denotes the concatenation of $s$ with itself $n$ times.
   $s^0 = \Lambda$.
Example:
   $(ab)^3 = ababab$
Languages

Given an alphabet $A$, a **language** is a set of strings over $A$.

Notation:

If $A$ is an alphabet, then the set of *all* strings over $A$ is denoted $A^*$. Some languages over $A$ are:

$\emptyset$, $\{\Lambda\}$, $A$, $A^*$

Example:

Let alphabet be $\{a, b\}$

$\{ab^n a \mid n \in \mathbb{N} \} = \{aa, aba, abba, abbba, \ldots\}$

Language Operations:

Let $L$ and $M$ be two languages.

The **product** of $L$ and $M$, denoted $LM$, is

$L M = \{ \text{st} \mid s \in L \text{ and } t \in M \}$

Example:

Let $L = \{a, b\}$ and $M = \{cc, ee\}$. Then…

$LM = \{acc, aee, bcc, bee\}$

$ML = \{cca, ccb, eea, eeb\}$
In-class Quiz:
What are the products \( L \emptyset \) and \( L \{ \Lambda \} \)?

In-class Quiz:
Solve for \( L \) in the equation
\[
\{ \Lambda, a, b \} L = \{ \Lambda, a, b, aa, ba, aba, bba \}
\]

Notation:
\[
\begin{align*}
    L^0 &= \{ \Lambda \} \\
    L^1 &= L \\
    L^2 &= LL \\
    L^n &= \{ s_1s_2...s_n \mid s_i \in L \}
\end{align*}
\]
The closure \( L^* \) is the set of all possible concatenations of strings in \( L \).
\[
L^* = L^0 \cup L^1 \cup ... \cup L^n \cup ...
\]

In-class quiz:
What are \( \{ \Lambda \}^* \) and \( \emptyset^* \)?
Example:
Examine the structure of an arbitrary string $x \in L^*(ML)^*$.

Approach: Use the definitions to write $x$ in terms of strings in $L$ and $M$.

Since $x \in L^*(ML)^*$, it follows that $x = uv$, where $u \in L^*$ and $v \in (ML)^*$.
Since $u \in L^*$, either $u = \Lambda$ or $u = s_1...s_n$ for some $n$ where $s_i \in L$.
Since $v \in (ML)^*$, either $v = \Lambda$ or $v = r_1t_1...r_kt_k$ for some $n$ where $r_i \in M$ and $t_i \in L$.
So $x$ must have one of four forms:

- $\Lambda$
- $s_1...s_n$
- $r_1t_1...r_kt_k$
- $s_1...s_n r_1t_1...r_kt_k$
Relations

A relation is a set of tuples. If \( R \) is a relation and \( (x_1, \ldots, x_n) \in R \), we write \( R(x_1, \ldots, x_n) \).

We can usually represent a relation as a subset of some cartesian product.

*Example:*

Let \( R = \{(0,0), (1,1), (4,2), (9,3), \ldots, (k^2,k), \ldots\} \)

\[ = \{(k^2,k) \mid k \in \mathbb{N}\} \]

We might call \( R \) the “is square of” relation on \( \mathbb{N} \).

Notice that \( R \subseteq \mathbb{N} \times \mathbb{N} \).

*Notation:*

If \( R \) is binary, we can use **infix** to represent pairs in \( R \).

For example, from the previous example, we have \( (9,3) \in R \)

So we can write:

\( R(9,3) \)

\( 9 \text{ R } 3 \)

\( 9 \text{ is-square-of } 3 \)
Relational Databases

A relational database is a relation where the indexes of a tuple have associated names, called attributes.

Example:
Let Students = \{ (x,y,z) | x is a Name, y is a Major, and z is Credits) \}

Who are the students majoring in CS?
\{ x | (x, “cs”, z) ∈ Students \}

Note: we need a way to tell values apart from variables: (x,cs,z)?

How many math majors are upper division students?
| | \{ x | (x, “math”, z) ∈ Students and z ≥ 90 \} |

What is the major of JohnSmith?
\{ y | (“JohnSmith”, y, z) ∈ Students \}

What is the Math departments database of names and credits?
\{ (x,y) | (x, “math”, z) ∈ Students \}

<table>
<thead>
<tr>
<th>Name</th>
<th>Major</th>
<th>Credits</th>
</tr>
</thead>
<tbody>
<tr>
<td>JohnSmith</td>
<td>cs</td>
<td>70</td>
</tr>
<tr>
<td>FredBrown</td>
<td>math</td>
<td>85</td>
</tr>
<tr>
<td>JackGreen</td>
<td>math</td>
<td>120</td>
</tr>
<tr>
<td>SueJones</td>
<td>cs</td>
<td>130</td>
</tr>
</tbody>
</table>
Counting Tuples (or strings or lists)

Product Rules:

$$|A \times B| = |A| \cdot |B|$$

$$|A^n| = |A|^n$$

Example: If $A = \{a,b\}$ and $B=\{1,2,3\}$ then

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3) \}$$

So $|A \times B| = |A| \cdot |B| = 2 \times 3 = 6$
Example:
Count the number of strings of length 8 over $A = \{a, b, c\}$ that begin with either $a$ or $c$ and have at least one $b$.

Solution: **Divide and conquer!**
Split the problem up into easier problems and combine the results.
Let $U$ be the universe = the set of strings over $A$ of length 8 that begin with either $a$ or $c$.
Let $B$ be the subset of $U$ consisting of strings with no $b$'s.
The set we want to count is then $U - B$.

Calculate the cardinality of $U - B$.
$$|U - B| = |U| - |U \cap B|$$
$$= |U| - |B|$$
since $B$ is a subset of $U$

What is the cardinality of $U$?
$$U = \{a, c\} \times A^7$$
$$|\{a, c\} \times A^7| = |\{a, c\}| \times |A^7| = |\{a, c\}| \times |A|^7 = 2(3)^7$$

What is the cardinality of $B$, the set of strings not containing $b$?
$$|\{a, c\}^8| = |\{a, c\}|^8 = 2^8$$

So the answer is:
$$|U - B| = |U| - |U \cap B|$$
$$= |U| - |B| = 2(3)^7 - 2^8 = 4118$$