Section 1.2: Sets

A set is a collection of things.
If $S$ is a set and $x$ is a member or element of $S$, we write
$x \in S$
If $x$ is not a member of $S$, we write
$x \notin S$
The set with elements $x_1, x_2, \ldots x_n$ is
$\{ x_1, x_2, \ldots x_n \}$
The empty set - A set with no elements
$\{\} \text{ or } \emptyset$
A singleton set – has only one element. Example: $\{a\}$

Useful Sets:
$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - the integers
$N = \{0, 1, 2, \ldots\}$ - the natural numbers, the non-negative integers
$Q$ – the rational numbers. Includes numbers like $\frac{1}{2}$ and 34.56
$R$ – the real number. Includes numbers like $\pi$ and $\sqrt{2}$
Set Equality

Two sets A and B are equal iff they have the same elements.

\[ A = B \]

Examples:

\[ \{a, b, c\} = \{c, b, a\} \quad \text{order does not matter} \]

\[ \{a, a, b, c\} = \{a, b, c\} \quad \text{repetitions are ignored, no repetitions} \]

Sets can also be described this way:

\[ \{ x \mid P \} = \text{the set of all elements that satisfy } P \]

where P is a property.

Example:

The set of all odd natural numbers:

\[ \{1, 3, 5, 7, \ldots\} = \{ x \mid x = 2k+1 \text{ for some } k \in \mathbb{N} \} \]

Subsets

Set A is a subset of B iff every element in A is also in B.

\[ A \subseteq B \]

Note, for any set S:

\[ S \subseteq S \]

\[ \emptyset \subseteq S \]

\[ \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \]


**Power Sets:**

The power set of a set $S$

$\text{power}(S)$

is the set of all subsets of $S$.

*Example:*

$\text{power}\left(\{a,b\}\right) = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \}$

**Comparing Sets:**

Let $A = \{ 2k+7 \mid k \in \mathbb{Z} \}$ and $B = \{ 4k+3 \mid k \in \mathbb{Z} \}$

*Question:* Is $A \subseteq B$?

*Answer:* No. For example, $9 \in A$ but $9 \notin B$.

*Question:* Is $B \subseteq A$?

*Answer:* Yes. Let $x \in B$. Then $x = 4k+3$ for some integer $k$.

We can write

$$x = 4k+3 = 4k-4+7 = 2(2k-2) + 7$$

Since $2k-2 \in \mathbb{Z}$, it follows that $x \in A$. Therefore $B \subseteq A$. QED
Equality in terms of subsets:

\[ A = B \iff A \subseteq B \text{ and } B \subseteq A \]

Example: Let \( A = \{ 2k+5 \mid k \in \mathbb{Z} \} \) and \( B = \{ 2k+3 \mid k \in \mathbb{Z} \} \)
Show that \( A = B \).

Proof: First show \( A \subseteq B \). Then show \( B \subseteq A \).

To show \( A \subseteq B \),
Let \( x \in A \). So \( x = 2k+5 \) for some integer \( k \).
We can write
\[ x = 2k+5 = 2k+2+3 = 2(k+1)+3 \]
Since \( k+1 \) is an integer, if follows that \( x \in B \). Therefore, \( A \subseteq B \).

In-class quiz:
Show \( B \subseteq A \).
Operations on Sets

Union
\[ A \cup B = \{ x \mid x \in A \text{ or } x \in B \}\]

Intersection
\[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \}\]

Difference
\[ A - B = \{ x \mid x \in A \text{ and } x \notin B \}\]

Symmetric Difference
\[ A \oplus B = \{ x \mid x \in A \text{ or } x \in B, \text{ but not both} \}\]
\[ = (A - B) \cup (B - A) \]
\[ = (A \cup B) - (A \cap B) \]

Universal Complement
Given a universe \( U \) and \( A \subseteq U \), we can write
\[ A' = U - A \]
Example: For each $n \in \mathbb{N}$ let $D_n = \{x \in \mathbb{N} \mid x \text{ divides } n \}$

So $D_n$ is the set of positive divisors of $n$.

Here are some expressions involving these sets:

- $D_0 = \{1, 2, 3, \ldots \} = \mathbb{N} - \{0\}$
- $D_5 = \{1, 5\}$
- $D_6 = \{1, 2, 3, 6\}$
- $D_9 = \{1, 3, 9\}$
- $D_5 \cup D_6 = \{1, 2, 3, 5, 6\}$
- $D_5 \cap D_6 = \{1\}$
- $D_9 - D_6 = \{9\}$
- $D_5 \oplus D_6 = \{2, 3, 5, 6\}$

Let $\mathbb{N}$ be the universe.

- $D_0' = \mathbb{N} - D_0 = \{0\}$
- $\{0\}' = D_0$

In-class quiz:

Draw a Venn Diagram for three sets $A$, $B$, $C$ with some areas shaded. Then find an expression to represent the shaded area.
Properties of Set Operations

Union and intersection are

**commutative**
\[ A \cup B = B \cup A \]
\[ A \cap B = B \cap A \]

**associative**
\[ (A \cup B) \cup C = A \cup (B \cup C) \]
\[ (A \cap B) \cap C = A \cap (B \cap C) \]

They **distribute** over each other
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]

**Absorption**
\[ A \cup (A \cap B) = A \cap (A \cup B) = A \]

**De Morgan’s Law**
\[ (A \cup B)' = A' \cap B' \]
\[ (A \cap B)' = A' \cup B' \]

**Set Algebra**
Given an expression over sets, you can rewrite it.
Counting Sets

The **cardinality** of a set $S$ is denoted by $|S|$. If the sets are finite, you can use these rules:

- **Inclusion-Exclusion or Union Rule:**
  \[|A \cup B| = |A| + |B| - |A \cap B|\]

- **Difference Rule:**
  \[|A - B| = |A| - |A \cap B|\]

**In-class Quiz:**
Find a rule for the union of 3 sets.

**In-Class Quiz:**
Three programs use a collection of CPUs in the following way:
There are 100 CPUs, shared among the programs. Each CPU may be used by 0,1,2, or all 3 programs. $A,B,C$ represent the sets of CPUs used by each program.

- $|A| = 20$, $|B| = 40$, $|C| = 60$, $|A \cap B| = 10$, $|A \cap C| = 8$, $|B \cap C| = 6$.

What value could $|A \cap B \cap C|$ have?

**Answer:**
100 $\geq |A \cup B \cup C| = 20 + 40 + 60 - 10 - 8 - 6 + |A \cap B \cap C|$
$|A \cap B \cap C| \leq 4$
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap D| + |A \cap B \cap C| \]
Bags (also called Multisets)

Like sets but can contain repeated elements.
[a,b,c,c,b] = [c,b,a,b,c]
Order is unimportant.

Union is defined by taking the maximum number of occurencences.
[a,b,c,c,c] ∪ [a,a,a,c,dd] = [a,a,a,b,c,c,c,d,d]

Intersection is defined by taking the minimum number of occurencences.
[a,b,c,c,c] ∩ [a,a,a,c,dd] = [a,c,]

In-class Quiz:
Let
A = [m,i,s,s,i,s,s,i,p,p,i]
B = [s,i,p,p,i,n,g]
What is A ∪ B?
What is A ∩ B?