Section 1.2: Sets

A **set** is a collection of things.
If S is a set and x is a **member** or element of S, we write

$x \in S$

If x is not a member of S, we write

$x \notin S$

The set with elements $x_1, x_2, \ldots x_n$ is

$\{ x_1, x_2, \ldots x_n \}$

The **empty set** - A set with no elements

{} or Ø

A **singleton** set – has only one element. Example: {a}

Useful Sets:

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - the **integers**

$\mathbb{N} = \{0, 1, 2, \ldots\}$ - the **natural** numbers, the non-negative integers

$\mathbb{Q}$ – the **rational** numbers. Includes numbers like ½ and 34.56

$\mathbb{R}$ – the **real** number. Includes numbers like π and $\sqrt{2}$

Set Equality

Two sets A and B are equal iff they have the same elements.

$A = B$

Examples:

$\{a, b, c\} = \{c, b, a\}$ order does not matter

$\{a, a, b, c\} = \{a, b, c\}$ repetitions are ignored, no repetitions

Sets can also be described this way:

$\{ x \mid P \}$ = the set of all elements that satisfy P

where P is a property.

Example:

The set of all odd natural numbers:

$\{1,3,5,7,\ldots\} = \{ x \mid x = 2k+1 \text{ for some } k \in \mathbb{N} \}$

Subsets

Set A is a subset of B iff every element in A is also in B.

$A \subseteq B$

Note, for any set S:

$S \subseteq S$

$\emptyset \subseteq S$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Power Sets:

The power set of a set S

power(S)

is the set of all subsets of S.

Example:

power({a,b}) = { Ø, {a}, {b}, {a,b} }

Comparing Sets:

Let A = { 2k+7 | k $\in \mathbb{Z}$ } and B = { 4k+3 | k $\in \mathbb{Z}$ }

Question: Is A $\subseteq$ B?

Answer: No. For example, 9 $\in$ A but 9 $\notin$ B.

Question: Is B $\subseteq$ A?

Answer: Yes. Let x $\in$ B. Then x = 4k+3 for some integer k.

We can write

x = 4k+3 = 4k-4+7 = 2(2k-2) + 7

Since 2k-2 $\in \mathbb{Z}$, it follows that x $\in$ A. Therefore B $\subseteq$ A. QED

Equality in terms of subsets:

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

Example: Let $A = \{ 2k+5 \mid k \in \mathbb{Z} \}$ and $B = \{ 2k+3 \mid k \in \mathbb{Z} \}$

Show that $A = B$.

Proof: First show $A \subseteq B$. Then show $B \subseteq A$.

To show $A \subseteq B$,

Let $x \in A$. So $x = 2k+5$ for some integer k.

We can write

$$x = 2k+5 = 2k+2+3 = 2(k+1)+3$$

Since k+1 is an integer, if follows that $x \in B$. Therefore, $A \subseteq B$.

In-class quiz:

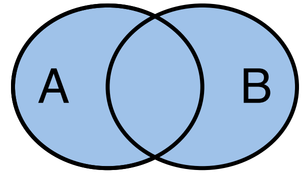
Show $B \subseteq A$.

Operations on Sets

Venn Diagrams

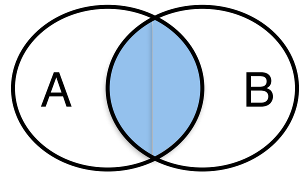
Union

$A \cup B = \{ x \mid x \in A$ **or** $x \in B \}$



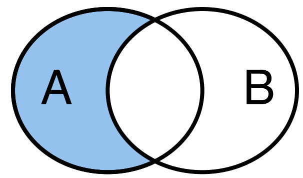
Intersection

$A \cap B = \{ x \mid x \in A$ **and** $x \in B \}$



Difference

$A - B = \{ x \mid x \in A$ **and** $x \notin B \}$

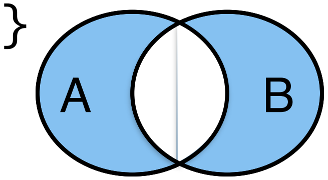


Symmetric Difference

$A \oplus B = \{x \mid x \in A$ or $x \in B$, but not both $\}$

$= (A - B) \cup (B - A)$

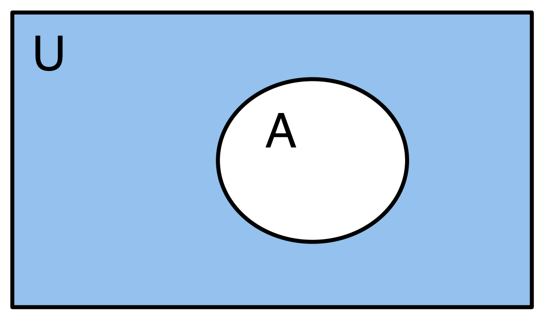
$= (A \cup B) - (A \cap B)$



Universal Complement

Given a universe U and $A \subseteq U$, we can write

$A' = U - A$



Example: For each $n \in \mathbb{N}$ let $D_n = \{x \in \mathbb{N} \mid x \text{ divides } n\}$

So D_n is the set of positive divisors of n.

Here are some expressions involving these sets:

$D_0 = \{1, 2, 3, \ldots\} = \mathbb{N} - \{0\}$

$D_5 = \{1, 5\}$

$D_6 = \{1, 2, 3, 6\}$

$D_9 = \{1, 3, 9\}$

$D_5 \cup D_6 = \{1, 2, 3, 5, 6\}$

$D_5 \cap D_6 = \{1\}$

$D_9 - D_6 = \{9\}$

$D_5 \oplus D_6 = \{2, 3, 5, 6\}$

Let N be the universe.

$D_0' = \mathbb{N} - D_0 = \{0\}$

$\{0\}' = D_0$

In-class quiz:

Draw a Venn Diagram for three sets A, B, C with some areas shaded.
Then find an expression to represent the shaded area.

Properties of Set Operations

Union and intersection are

commutative

$A \cup B = B \cup A$

$A \cap B = B \cap A$

associative

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

They **distribute** over each other

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption

$A \cup (A \cap B) = A \cap (A \cup B) = A$

De Morgan's Law

$(A \cup B)' = A' \cap B'$

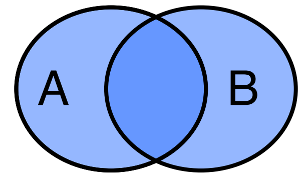
$(A \cap B)' = A' \cup B'$

Set Algebra

Given an expression over sets, you can rewrite it.

Counting Sets

The **cardinality** of a set S is denoted by $|S|$.
If the sets are finite, you can use these rules:



Inclusion-Exclusion or Union Rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Difference Rule:

$$|A - B| = |A| - |A \cap B|$$

In-class Quiz:

Find a rule for the union of 3 sets.

In-Class Quiz:

Three programs use a collection of CPUs in the following way:
There are 100 CPUs, shared among the programs.
Each CPU may be used by 0,1,2, or all 3 programs.
A,B,C represent the sets of CPUs used by each program.

$$|A|=20,\ |B|=40,\ |C|=60,\ |A \cap B|=10,\ |A \cap C|=8,\ |B \cap C|=6.$$

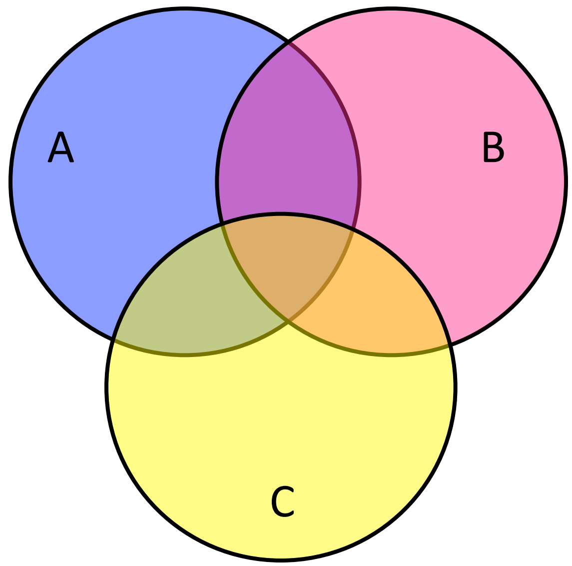
What value could $|A \cap B \cap C|$ have?

Answer:

$$100 \geq |A \cup B \cup C| = 20 + 40 + 60 - 10 - 8 - 6 + |A \cap B \cap C|$$

$$|A \cap B \cap C| \leq 4$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap D| + |A \cap B \cap C|$$



Bags (also called **Multisets**)

Like sets but can contain repeated elements.

[a,b,c,c,b] = [c,b,a,b,c]

Order is unimportant.

Union is defined by taking the maximum number of occurencences.

[a,b,c,c,c] ∪ [a,a,a,c,dd] = [a,a,a,b,c,c,c,d,d]

Intersection is defined by taking the minimum number of occurencences.

[a,b,c,c,c] ∩ [a,a,a,c,dd] = [a,c,]

In-class Quiz:

Let

A = [m,i,s,s,i,s,s,i,p,p,i]

B = [s,i,p,p,i,n,g]

What is A ∪ B?

What is A ∩ B?