We want to design a low-pass filter (LPF). Many tables exist that give the transfer function for "normalized" analog filters. For example, one such normalized analog LPF is

$$H(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1}$$

Normalized filters have a cutoff frequency of 1Hz (i.e., $\omega = 1$).

We want our LPF to have a cutoff frequency of 150Hz and a sample frequency of 1.28KHz. We will use the *impulse invariant method.*

First we must frequency scale the normalized analog filter transfer function to get the correct cutoff frequency $\omega_c = 2\pi(150) = 942.4778$.

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1} \\ &= \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \\ &= \frac{A}{s - p_1} + \frac{B}{s - p_2} \end{aligned}$$

NOTE: The impulse invariant method only will work with an all-pole analog transfer function.

The two poles are at p_1 = -666.4324(1+j) and p_2 = -666.4324(1-j). The partial fraction expansion yields

$$A = -\frac{\omega_c}{\sqrt{2}} j$$

$$B = +\frac{\omega_c}{\sqrt{2}} j$$

Now, we know

$$\frac{1}{s+a} \leftrightarrow \frac{1}{1 - e^{aT} z^{-1}}$$

Where $T = 1/f_s$ is the sample rate. This gives a filter transfer function

$$H(z) = \frac{A}{1 - e^{p_1 T} z^{-1}} + \frac{B}{1 - e^{p_2 T} z^{-1}}$$
$$= \frac{393.9264 z^{-1}}{1 - 1.0308 z^{-1} + 0.3530 z^{-2}}$$

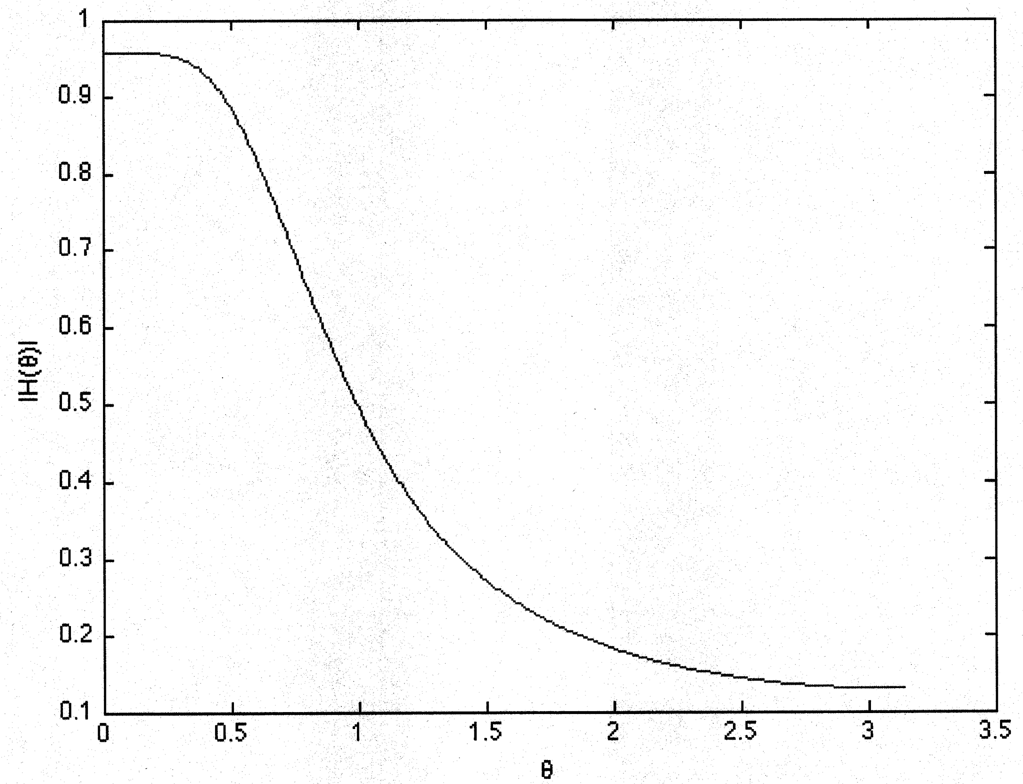
after substituting in the sample rate T, the A and B coefficients and the poles p_1 and p_2.

Then substitute $z = e^{j\theta}$ to get

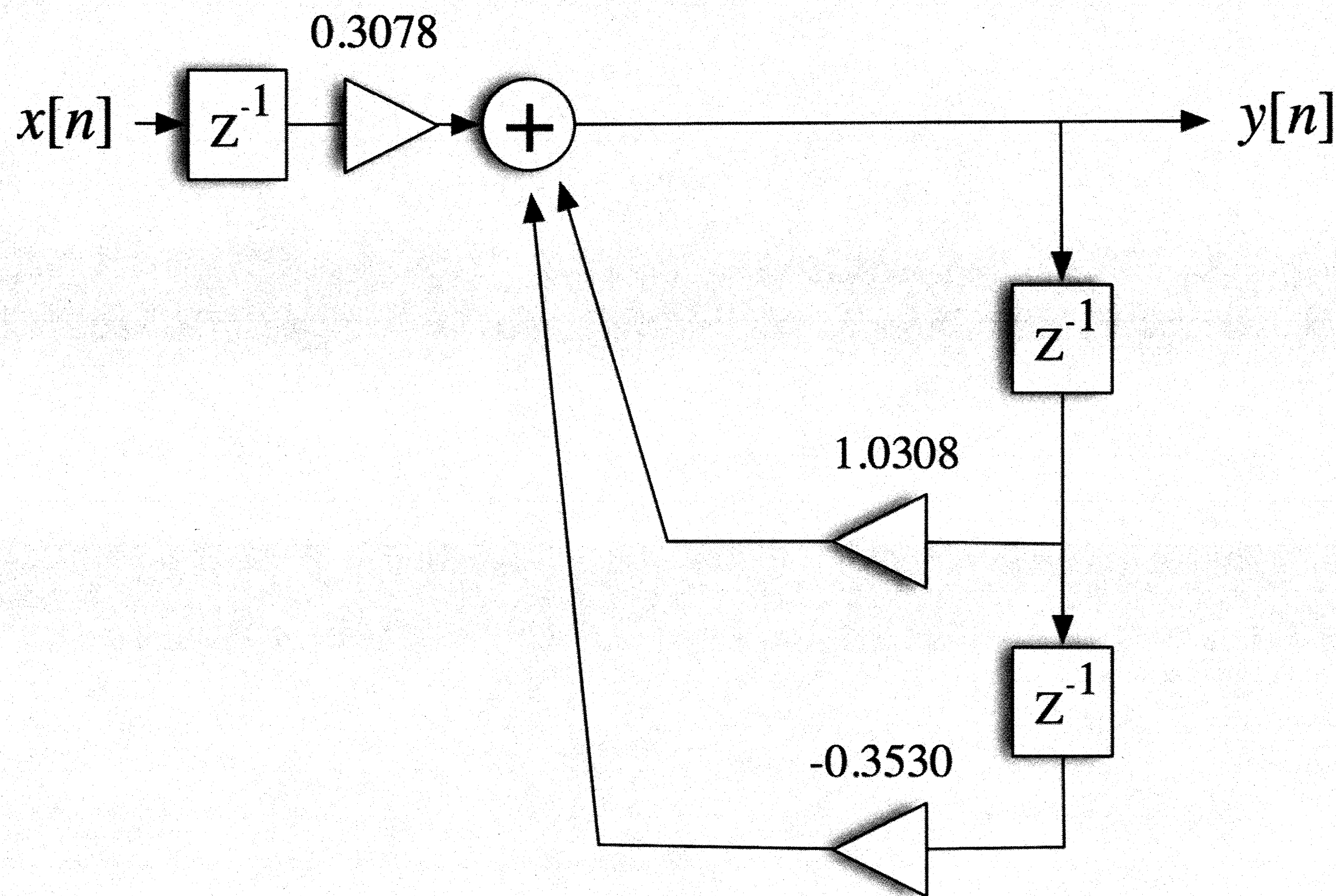
$$H\left(e^{j\theta}\right) = H(\theta) = \frac{393.9264 e^{-j\theta}}{1 - 1.0308 e^{-j\theta} + 0.3530 e^{-j2\theta}}$$

Notice at $\theta = 0$, $|H(\theta)| \approx 1223$ which is close to the sample frequency in Hz (1280).

This large gain value is typical of IIR filters and can possibly cause overflow problems. So it is common practice to multiply the filter gain by the sample rate (T).



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
|H(θ)|
0
0.5
1
1.5
2
2.5
3
3.5
θ



x[n]
z-1
0.3078
+
y[n]
z-1
1.0308
z-1
-0.3530

$$H(z) = \frac{0.3078z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

Since $H(z) = \frac{Y(z)}{X(z)}$ we have

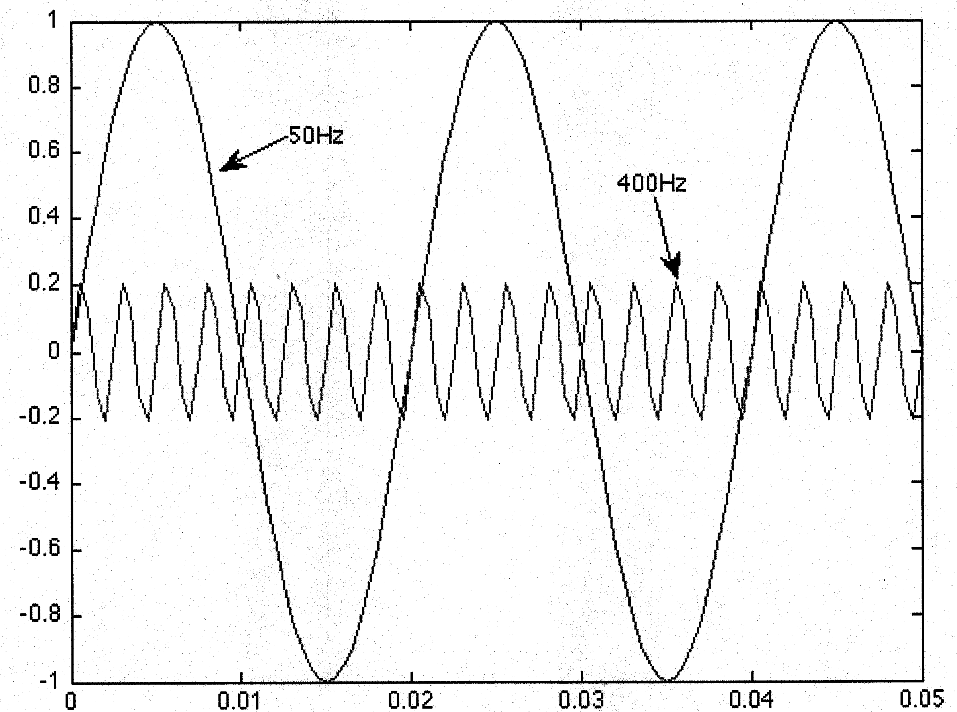
$$Y(z) - 1.0308z^{-1}Y(z) + 0.3530Y(z) = 0.3078z^{-1}X(z)$$

The inverse Z transform becomes

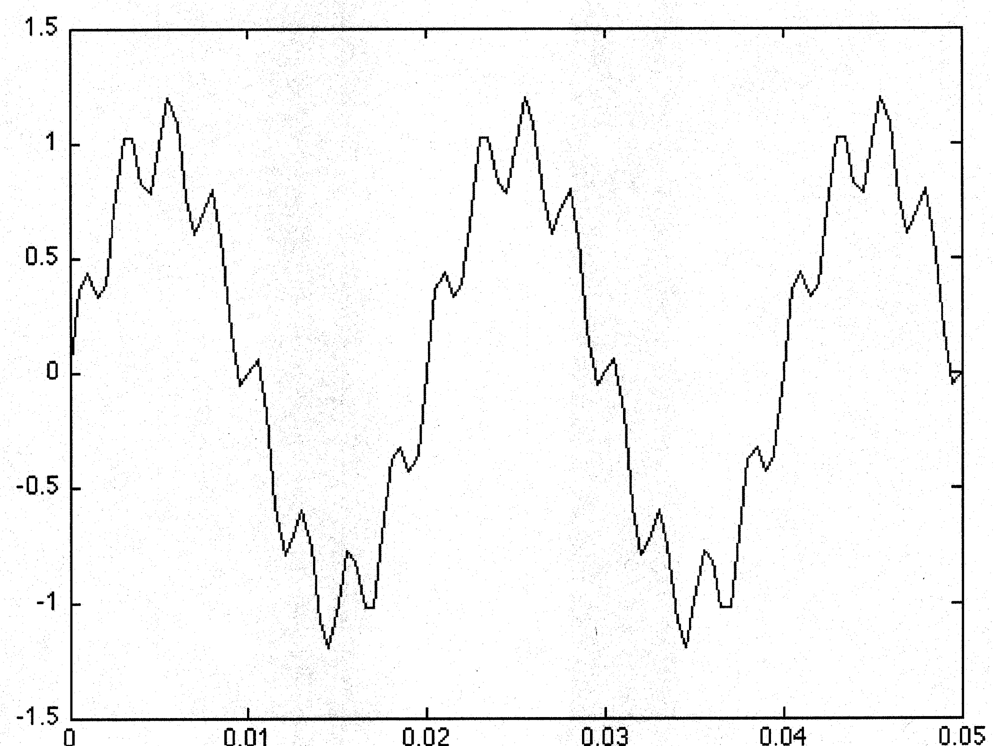
$$y[n] \; - \; 1.0308y[n-1] \; + \; 0.3530y[n-2] \; = \; 0.3078x[n-1]$$

or,

$$y[n] \; = \; 1.0308y[n-1] \; - \; 0.3530y[n-2] \; + \; 0.3078x[n-1]$$



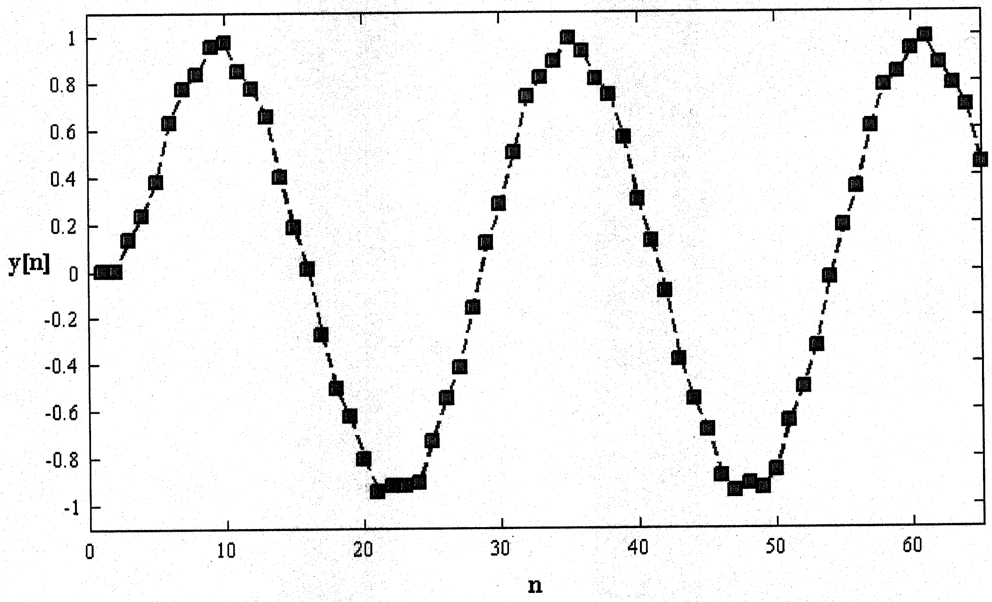
$$x(t) = \sin(2\pi(50)t) + \sin(2\pi(400)t)$$



$$x(nT) = x[n] = \sin(\underbrace{2\pi(50)nT}_{=n\theta_1}) + 0.22\sin(\underbrace{2\pi(400)nT}_{=n\theta_2})$$

$$= \sin(n\theta_1) + 0.22\sin(n\theta_2)$$

Plugging into the difference equation gives the output $y[n] = y[nT]$.



y[n]
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
10
20
30
40
50
60
n