1. Consider a LTI system with input $x[n]$ and an impulse response

$$h[n]=\begin{cases}(0.5)^n & n=0,1,2\\ 0 & \text{otherwise}\end{cases}$$

a) (3 points) Find $H(z)$.

b) (3 points) Find the unit step response (i.e., when $x[n]=u[n]$)

c) (2 points) What is the region of convergence?

d) (2 points) Where are the poles located? (Tell me how many and their values)

a) $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = 1 + 0.5z^{-1} + 0.25z^{-2}$

(alternate form)

$= \left(1 + 0.5z^{-1} + 0.25z^{-2}\right)\frac{z^2}{z^2}$

$= \frac{z^2 + 0.5z + 0.25}{z^2}$

b) $y[n] = \sum_{k=0}^{n} h[k]$

$y[0] = h[0] = 1$

$y[1] = h[0] + h[1] = 1.5$

$y[k] = h[0] + h[1] + h[2] = 1.75 \quad \forall k \geq 2$

c) $|z| > 0$

d) 2 poles, both at $z=0$

2. An aperiodic signal $x(t)$ is sampled between $0 \le t \le 5$ seconds. Samples are taken every 20 milliseconds to construct an N-point DFT.

a) (3 points) What is the sample frequency in Hz?

b) (3 points) The DFT harmonic $C[4]$ corresponds to what frequency in Hz?

c) (4 points) You want to zero-pad this signal so the number of samples is an integer power of two. What is the minimum number of zeros that you would add to the sample sequence?

a) 50 Hz

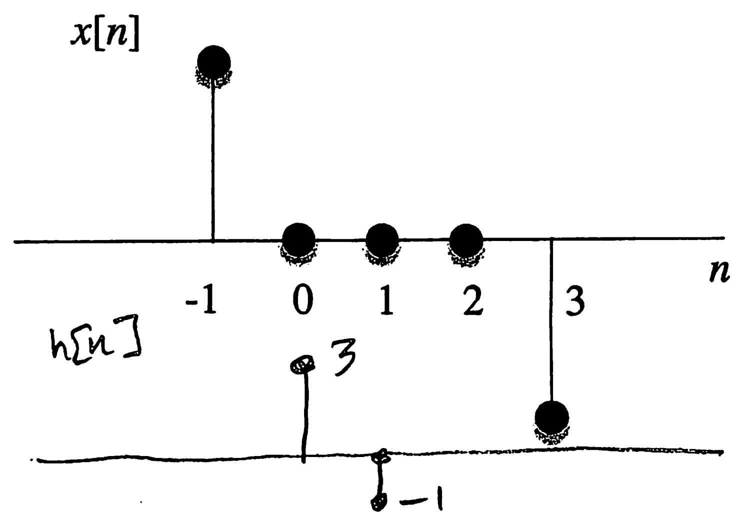
b) 50 samples/s × 5s = 250 samples = N

$$\Delta f = \frac{f_s}{N} = \frac{50}{250} = 0.2\,Hz$$

∴ C[4] corresponds to $(4)(0.2) = 0.8\,Hz$

c) 6 zeros

3. (10 points) Consider



Suppose $h[n] = 3\delta[n] - \delta[n-1]$. Find $y[n]$ via convolution for $n = 0, 1, 2, 3, 4, 5$.

$h[0] = 3$, $h[1] = -1$ $\quad h[k] = 0 \ \forall k \neq 0, 1$

$y[0] = h[0]x[0] + h[1]x[-1] = 0 + (-1) = -1$

$y[1] = h[0]x[1] + h[1]x[0] = 0 + 0 = 0$

$y[2] = h[0]x[2] + h[1]x[1] = 0 + 0 = 0$

$y[3] = h[0]x[3] + h[1]x[2] = -3 + 0 = -3$

$y[4] = h[0]x[4] + h[1]x[3] = 0 + 1 = 1$

$y[5] = h[0]x[5] + h[1]x[4] = 0 + 0 = 0$

4. A LTI system with input $x[n]$ and output response $y[n]$ is described by the difference equation

$$y[n]+2y[n-2]=x[n]$$

(a) (3 points) Find $H(z)$

(b) (4 points) Find the impulse response. (Give values for all n.)

(c) (3 points) Does this system have a DTFT? (You must justify your answer to get any points on this portion of the problem.)

a) $Y(z) + 2z^{-2}Y(z) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+2z^{-2}} = \frac{z^2}{z^2+2}$$

b) $y[n] = x[n] - 2y[n-2]$

$y[0] = x[0] = 1$

$y[1] = x[1] - 2y[-1] = 0$

$y[2] = x[2] - 2y[0] = -2$

$y[3] = x[3] - 2y[1] = 0$

$y[4] = x[4] - 2y[2] = +4$

$y[5] = x[5] - 2y[3] = 0$

$y[6] = x[6] - 2y[4] = -8$

$\vdots$

$\Rightarrow$ for $K = 0, 1, 2, 3, \ldots$

$y[2K] = (j)^{2K} \cdot 2^K$

or, $y[2K] = (-2)^K$

$y[n] = 0$ otherwise

c) ROC does not contain unit circle (ROC: $|z| > \sqrt{2}$)

So DTFT does not exist

5. Let $X(\omega) = \begin{cases} \cos(\omega) & |\omega| \le \pi/2 \\ 0 & \text{otherwise} \end{cases}$

(a) (6 points) Find $x(t)$

(b) (4 points) Find $X(\omega)$ for $x(t+3)$.

a) $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\omega \, e^{j\omega t}\, d\omega$$

from the integral tables,

$$= \left(\frac{1}{2\pi}\right) e^{j\omega t}\left(\frac{\sin\omega + jt\cos\omega}{1-t^2}\right)\Bigg|_{-\pi/2}^{\omega=\pi/2}$$

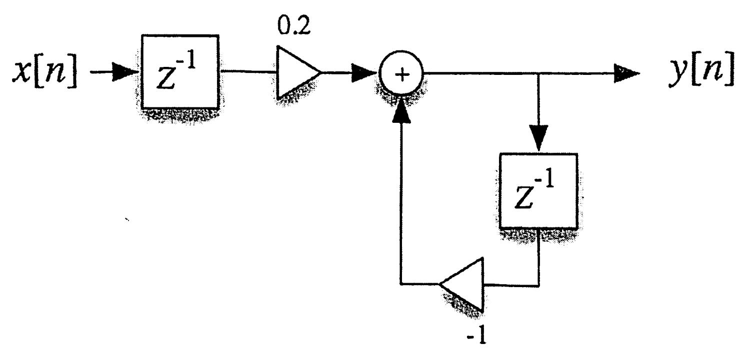
$$= \left[e^{j\frac{\pi}{2}t}\left(\frac{1}{1-t^2}\right) - e^{-j\frac{\pi}{2}t}\left(\frac{-1}{1-t^2}\right)\right]\cdot\left(\frac{1}{2\pi}\right)$$

$$= \left(\frac{1}{2\pi}\right)\left[\frac{e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}}{1-t^2}\right]$$ ← (full credit if you get this far)

$$= \left(\frac{1}{\pi}\right)\cdot\frac{\cos\frac{\pi}{2}t}{1-t^2}$$

b) $$X(\omega) = \begin{cases} e^{j3\omega}\cos(\omega) & |\omega| < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

6.



(a) (4 points) Find $H(\theta)$.

(b) (2 points) Assuming the sample frequency is 1.0 KHz, what frequency (in Hz) corresponds to $\theta = \pi/4$?

(c) (4 points) Does this system have a DTFT? (You must justify your answer to receive credit for this part)

a) $y[n] = 0.2\,x[n-1] - y[n-1]$

$y[n] + y[n-1] = 0.2\,x[n-1]$

$Y(z)\left[1+z^{-1}\right] = 0.2 z^{-1} X(z)$

$$H(z) = \frac{0.2 z^{-1}}{1+z^{-1}} = \frac{0.2}{z+1}$$

$$\therefore\ H(\theta) = \frac{0.2}{1+e^{j\theta}}$$

b) $\frac{f_s}{2\pi} = \frac{f}{\pi/4} \Rightarrow f = \frac{f_s}{8} = 125\ \text{Hz}$

c) converges for $|z| > 1$

does not contain unit circle

$\therefore$ No DTFT