Crank Nicolson Solution to the Heat Equation

ME 448/548 Notes

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Overview

1. Use finite approximations to $\partial u / \partial t$ and $\partial^2 u / \partial x^2$: same components used in FTCS and BTCS.

2. Average contribution from $t_k$ and $t_{k+1}$ in the approximation of $\partial^2 u / \partial x^2$.

3. Computational formula is (still) implicit: all $u_{i}^{k+1}$ must be solved simultaneously. Information from $u_{i-1}^{k}$, $u_{i}^{k}$, and $u_{i+1}^{k}$ are used at each time step in the computation of $u_{i}^{k+1}$.

4. Solution is not significantly more complex than BTCS.

5. Like BTCS, the Crank-Nicolson methods is unconditionally stable for the heat equation.

6. Truncation error is $O(\Delta t^2) + O(\Delta x^2)$. Time accuracy is better than BTCS or FTCS.
The $\theta$ Method

Evaluate the diffusion operator $\partial^2 u / \partial x^2$ at both time steps $t_{k+1}$ and time step $t_k$, and use a weighted average

$$
\frac{u_i^{k+1} - u_i^k}{\Delta t} = \alpha \theta \left[ \frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{\Delta x^2} \right] + \alpha (1 - \theta) \left[ \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{\Delta x^2} \right]
$$

(1)

where

$$0 \leq \theta \leq 1$$

$$\theta = 0 \iff \text{FTCS}$$

$$\theta = 1 \iff \text{BTCS}$$
Crank-Nicolson Computational Molecule

Solution is known for these nodes.

Crank-Nicolson scheme requires simultaneous calculation of $u$ at all nodes on the $k+1$ mesh line.

$t = 0, k = 1$

$x = 0$

$x = L$

$i = 1, i-1, i, i+1, n_x$

Cramped-Nicolson scheme requires simultaneous calculation of $u$ at all nodes on the $k+1$ mesh line.

Solution is known for these nodes.
Compare Computational Molecules

FTCS

\[
\begin{align*}
  &\text{(i)} \quad k+1 \\
  &\text{(k)} \\
  &\text{(i-1)} \quad i \quad i+1
\end{align*}
\]

BTCS

\[
\begin{align*}
  &\text{(i)} \quad k+1 \\
  &\text{(k)} \\
  &\text{(i-1)} \quad i \quad i+1
\end{align*}
\]

Crank-Nicolson

\[
\begin{align*}
  &\text{(i)} \quad k+1 \\
  &\text{(k)} \\
  &\text{(i-1)} \quad i \quad i+1
\end{align*}
\]
Crank Nicolson Approximation to the Heat Equation

\[- \frac{\alpha}{2\Delta x^2} u_{i-1}^{k+1} + \left( \frac{1}{\Delta t} + \frac{\alpha}{\Delta x^2} \right) u_i^{k+1} - \frac{\alpha}{2\Delta x^2} u_{i+1}^{k+1} = \]

\[\frac{\alpha}{2\Delta x^2} u_{i-1}^k + \left( \frac{1}{\Delta t} - \frac{\alpha}{\Delta x^2} \right) u_i^k + \frac{\alpha}{2\Delta x^2} u_{i+1}^k \quad (2)\]
Crank-Nicolson System of Equations

The system of equations has the same structure as BTCS

\[
\begin{bmatrix}
  a_1 & b_1 & 0 & 0 & 0 & 0 \\
  c_2 & a_2 & b_2 & 0 & 0 & 0 \\
  0 & c_3 & a_3 & b_3 & 0 & 0 \\
  0 & 0 & \ldots & \ldots & \ldots & 0 \\
  0 & 0 & 0 & c_{n-1} & a_{n-1} & b_{n-1} \\
  0 & 0 & 0 & 0 & c_{nx} & a_{nx}
\end{bmatrix}
\begin{bmatrix}
  u_1^{k+1} \\
  u_2^{k+1} \\
  u_3^{k+1} \\
  \vdots \\
  u_{n-1}^{k+1} \\
  u_n^{k+1}
\end{bmatrix}
= \begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
  \vdots \\
  d_{n-1} \\
  d_n
\end{bmatrix}
\]  

(3)

where the coefficients of the interior nodes \((i = 2, 3, \ldots, N - 1)\) are

\[
a_i = 1/\Delta t + \alpha/\Delta x^2 = 1/\Delta t - (b_i + c_i),
\]

\[
b_i = c_i = -\alpha/(2\Delta x^2),
\]

\[
d_i = -c_i u_{i-1}^k + (1/\Delta t + b_i + c_i) u_i^k - b_i u_{i+1}^k.
\]
% --- Coefficients of the tridiagonal system
b = (-alfa/2/dx^2)*ones(nx,1);  % Super diagonal: coefficients of u(i+1)
c = b;                         % Subdiagonal:  coefficients of u(i-1)
a = (1/dt)*ones(nx,1) - (b+c);  % Main Diagonal: coefficients of u(i)
at = (1/dt + b + c);           % Coefficient of u_i^k on RHS
a(1) = 1;  b(1) = 0;           % Fix coefficients of boundary nodes
a(end) = 1;  c(end) = 0;
[e,f] = tridiagLU(a,b,c);       % Save LU factorization

% --- Assign IC and save BC values in ub. IC creates u vector
x = linspace(0,L,nx)';  u = sin(pi*x/L);  ub = [0 0];

% --- Loop over time steps
for k=2:nt
% --- Update RHS for all equations, including those on boundary
d = - [0;  c(2:end-1).*u(1:end-2); 0] ... 
  + [ub(1);  at(2:end-1).*u(2:end-1); ub(2)] ... 
  - [0;    b(2:end-1).*u(3:end);  0];
u = tridiagLUsolve(e,f,b,d);         % Solve the system
end
Convergence of FTCS, BTCS and CN

Reduce both $dx$ and $dt$ within the FTCS stability limit

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Spatial</th>
<th>Temporal</th>
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<tbody>
<tr>
<td>FTCS</td>
<td>$\Delta x^2$</td>
<td>$\Delta t$</td>
</tr>
<tr>
<td>BTCS</td>
<td>$\Delta x^2$</td>
<td>$\Delta t$</td>
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<tr>
<td>C-N</td>
<td>$\Delta x^2$</td>
<td>$\Delta t^2$</td>
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<table>
<thead>
<tr>
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<th>FTCS</th>
<th>BTCS</th>
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Reduce $dt$ while holding $dx = 9.775171e-04$ (L=1.0, nx=1024) constant

<table>
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Convergence of FTCS, BTCS and CN

In the plot of truncation error versus $\Delta t$ (right hand plot), there is an irregularity at $\Delta t \sim 3.9 \times 10^{-3}$. At that level of $\Delta t$, and for the chosen $\Delta x$ (which is constant), the truncation error due to $\Delta x$ is no longer negligible. Further reductions in $\Delta t$ alone will not reduce the total truncation error.
Summary for the Crank-Nicolson Scheme

- The Crank-Nicolson method is more accurate than FTCS or BTCS. Although all three methods have the same spatial truncation error ($\Delta x^2$), the better temporal truncation error for the Crank-Nicolson method is a big advantage.
- Like BTCS, the Crank-Nicolson scheme is unconditionally stable for the heat equation.
- Like BTCS, a system of equations for the unknown $u_i^k$ must be solved at each time step. The tridiagonal solver for the 1D heat equation obtains an efficient solution of the system of equations.
- The Crank-Nicolson scheme is recommended over FTCS and BTCS.