An Introduction to Turbulence Modeling for CFD

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Turbulence is a Hard Problem

- Unsteady
- Many length scales
- Energy transfer between scales:
  - Large eddies break up into small eddies
- Steep gradients near the wall
Engineering Model: Flow is “Steady-in-the-Mean” (1)
Engineering Model: Flow is “Steady-in-the-Mean” (2)

Reality:

Turbulent flows are unsteady: fluctuations at a point are caused by convection of eddies of many sizes. As eddies move through the flow the velocity field changes in complex ways at a fixed point in space.

Turbulent flows have structures – blobs of fluid that move and then break up.
Engineering Model: Flow is “Steady-in-the-Mean” (3)

Engineering Model:

When measured with a “slow” sensor (e.g. Pitot tube) the velocity at a point is apparently steady. For basic engineering analysis we treat flow variables (velocity components, pressure, temperature) as time averages (or ensemble averages). These averages are steady (ignoring ensemble averaging of periodic flows).
Turbulent eddies enhance mixing.

- Transport in turbulent flow is much greater than in laminar flow: e.g. pollutants spread more rapidly in a turbulent flow than a laminar flow.

- As a result of enhanced local transport, mean profiles tend to be more uniform except near walls.

- Near walls, gradients of velocity, temperature (and other scalars (e.g., chemical concentration) are steep.

  Visualize two-layer: high viscosity in core of pipe and low viscosity near the wall – Cartoon view only!
Engineering Model: Enhanced Transport (2)

Turbulence models are a way to account for enhanced mixing while treating the flow as steady-in-the-mean

- *Apparent* effect of turbulence is to increase the *effective* viscosity, thermal conductivity, and diffusivity.

- Enhanced transport coefficients are properties of the flow, not real thermophysical transport coefficients.

- Most commonly used turbulence models provide a way to compute the effective transport coefficients, e.g. the turbulence viscosity.
Computational Models (1)

1. Direct Numerical Simulation (DNS)

- No modeling of turbulence
- Resolve all time scales and spatial scales
- Useful for fundamental research
- Don’t scale to highest Re and hard to implement in complex geometries

Rodi\(^1\) cites a DNS simulation in 2015

With increasing computer power, channel-flow simulations with ever-increasing \( R \) were carried out over the years, and at the time of writing, the largest calculation is that of Lee and Moser (2015) at \( R = 125,000 \). This employed \( 242 \times 10^9 \) grid points and ran for several months on the Peta-FLOP/S computer Mira.

\(^1\)Journal of Hydraulic Engineering, 2017, 143(5)
World’s Fastest computers as of November 2018

See https://www.top500.org/lists/2018/11/. Mira is now #21 with $R_{\text{max}} = 0.06 R_{\text{max}}(\text{Summit})$

Note: the average US home uses $\approx 1$ kW.
Trend in the Top 500

Projected Performance Development

Computational Models (2)

2. Large Eddy Simulation (LES)

- Model the small scale (sub-grid) turbulence
- Larger, energy-containing eddies are resolved in unsteady flow
- Transient data needs to be averaged to obtain engineering quantities
- Useful for rigorous engineering analysis
- StarCCM+ can do LES and DES (Detached Eddy Simulation)
3. Reynolds-Averaged Navier Stokes (RANS)

- Turbulence effects are replaced by enhanced mixing – eddy viscosity
- Standard approach for basic engineering computations
- Extremely limited ability to capture complex structure
Reynolds Averaged Navier Stokes
RANS
Reynolds Averaged Navier-Stokes (RANS)

- Average the Navier-Stokes equations in attempt to solve equations that are steady in the mean.

- Averaging creates additional unknowns: the Reynolds stresses

- Use a turbulence model to compute the Reynolds stresses.
  - Simplest models are based on the Boussinesq eddy viscosity
  - Additional equations are necessary to compute the eddy viscosity
  - The $k - \epsilon$ model is the standard engineering model
Reynolds Decomposition (1)

Instantaneous velocity is

\[ \mathbf{u}(x, y, z, t) = \hat{e}_x u(x, y, z, t) + \hat{e}_y v(x, y, z, t) + \hat{e}_z w(x, y, z, t) \]

Decompose each component into a mean and fluctuating component

\[ u = U + u' \quad v = V + v' \quad w = W + w' \]
Reynolds Decomposition (2)

Reynolds averaging rules

1. \( \bar{f} + g = \bar{f} + \bar{g} \)

2. \( a \bar{f} = \bar{a f} \quad (a \text{ is a constant}) \)

3. \( \frac{\partial f}{\partial s} = \frac{\partial \bar{f}}{\partial s} \)

4. \( \bar{f}g = \bar{f} \bar{g} \)
Reynolds Decomposition (3)

Substitute into the Navier-Stokes equations and average. Consider one of the convection terms in the $x$-direction momentum equation

\[
\frac{\partial}{\partial x}(\rho uu) = \frac{\partial}{\partial x} \left[ \rho(U + u')(U + u') \right] \quad \text{rule 3}
\]

\[
= \frac{\partial}{\partial x} \left[ \rho \left( \overline{UU} + 2\overline{Uu'} + \overline{u'u'} \right) \right] \quad \text{rule 2, with } \rho = \text{constant}
\]

\[
= \frac{\partial}{\partial x} \left( \rho \overline{UU} + \overline{u'^2} \right) \quad \text{by definition of } U, \text{ and } \overline{Uu'} = \overline{Uu'} = 0
\]

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Reynolds Decomposition (4)

The averaging process has the important property that although $\overline{u'} = 0$,

$$\overline{u'u'} \neq 0$$

Terms like $\overline{u_i'u_j'}$ are called Reynolds stresses, and arise from the process of Reynolds averaging of the momentum equations.
Reynolds Decomposition (5)

There are six Reynolds stresses, which can be represented by the following matrix.

\[
\begin{bmatrix}
    u'u' & u'v' & u'w' \\
    u'v' & v'v' & v'w' \\
    u'w' & v'w' & w'w'
\end{bmatrix}
\]

The matrix is symmetric because \( u'u' = v'u' \) and \( v'w' = w'v' \). The Reynolds stresses are field variables, i.e., each term in the matrix is a function of space in the flow.
Another important quantity is the turbulence kinetic energy

\[ k = \frac{1}{2} u'_k u'_k = \frac{1}{2} (u'u' + v'v' + w'w') \]  

(1)

If the turbulence is *isotropic*, then \( u'u' = v'v' = w'w' \), and\(^2\)

\[ k = \frac{3}{2} u'u' \]  

(2)

\(^2\)Equality of normal stresses is a consequence of isotropy, not the definition of it.
Turbulence Intensity (1)

Turbulence Intensity is a measure of the velocity associated with the fluctuations in the flow relative to the mean flow. We start by defining the turbulence intensity in each coordinate direction

\[ TI_x = \frac{\sqrt{u'u'}}{U_{x,0}}, \quad TI_y = \frac{\sqrt{v'v'}}{U_{y,0}}, \quad TI_z = \frac{\sqrt{w'w'}}{U_{z,0}}, \quad (3) \]

where \( U_0 \) is a representative length scale.

The total turbulence intensity is

\[ TI = \sqrt{\frac{1}{3} (u'u' + v'v' + w'w')} \frac{1}{U_0} \quad (4) \]
Turbulence Intensity (2)

If the turbulence is isotropic, then from

$$TI = \sqrt{\frac{1}{3} (u'u' + u'u' + u'u')} = \sqrt{\frac{u'u'}{U_0}}$$  \hspace{1cm} (5)

and from Equation (2),

$$TI = \sqrt{\frac{2k}{3U_0}}$$  \hspace{1cm} (6)

thus, another interpretation of turbulence intensity is that it is a measure of (the square root of) local turbulence kinetic energy.
Turbulence Models
Kinds of turbulence models

1. Large eddy simulation
   a. Resolve unsteady large scale motions
   b. Model small scale (sub-grid scale) motions

2. Reynolds stress models
   a. Treat flow as steady (no large scale motions)
   b. Solve model equations for the Reynolds stresses

3. Two-equation models
   a. Treat flow as steady (no large scale motions)
   b. Use Boussinesq eddy viscosity to represent Reynolds stresses
   c. Solve model equations to compute eddy viscosity
Boussinesq model (1)

The viscous stress tensor is

$$\tau_{ij} = \mu \left[ \left( \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \frac{\partial U_k}{\partial x_k} \right] \tag{7}$$

Assume (wish, hope) that the Reynolds stresses have the same form as the viscous stresses.

$$-\rho u_i' u_j' = \mu_t \left[ \left( \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \frac{\partial U_k}{\partial x_k} \right] - \frac{2}{3} \rho \delta_{ij} k \tag{8}$$

where $k$ is the turbulence kinetic energy.
Boussinesq model (2)

In practice, the Boussinesq model involves replacing the viscosity in the momentum equations with an effective viscosity

\[ \mu_{\text{eff}} = \mu + \mu_t \]

where \( \mu \) is the physical or molecular viscosity, and \( \mu_t \) is a turbulence viscosity simulates the enhanced mixing due to turbulence.

General observations

- \( \mu_t \) is a point value. \( \mu_t \) has no direction, so therefore the Boussinesq model assumes that the turbulence is isotropic

- \( \mu_t \) needs to be estimated by some other model
### Prandtl’s Mixing Length Hypothesis

Prandtl made the direct analogy with the molecular model of viscosity from the kinetic theory of gases.

<table>
<thead>
<tr>
<th>molecular viscosity of gases</th>
<th>turbulence viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = \frac{1}{3} \rho \ell_f V_m$</td>
<td>$\mu_t = \rho \ell_m V_t$</td>
</tr>
<tr>
<td>$\ell_f = \text{mean free path}$</td>
<td>$\ell_m = \text{mixing length}$</td>
</tr>
<tr>
<td>$V_m = \text{velocity of molecules}$</td>
<td>$V_t = \text{velocity scale of turbulence}$</td>
</tr>
</tbody>
</table>

The mixing length is taken as an appropriate length scale for the flow.

The mixing length can vary in proportion to distance from a wall.
Estimate of Eddy Viscosity for Pipe Flow (1)

We can estimate the magnitude of the eddy viscosity in a pipe. Prandtl’s model is

\[ \mu_t \sim \rho V_t \ell_m \]

Turbulent fluctuations are small compared to the mean velocity in the pipe, say,

\[ 0.01 \leq \frac{u'}{U} \leq 0.15 \]

To estimate \( \mu_t \), take \( u'/U \sim 0.1 \) and \( V_t \sim u' \) so that

\[ V_t \sim 0.1U \]
Estimate of Eddy Viscosity for Pipe Flow (2)

To make a somewhat arbitrary choice of a single length scale that characterizes the turbulent mixing, take

\[ \ell_m \sim \frac{D}{4} \]

Combining the preceding expressions gives the following estimate of turbulence viscosity

\[ \mu_t = \rho(0.1U)(0.25D) = 0.025\rho UD \]

From the estimate of \( \mu_t \) we can compute an effective Reynolds number

\[ \text{Re}_{\text{eff}} = \frac{\rho UD}{\mu_t} = \frac{\rho UD}{0.025\rho UD} = 40 \]
The $k-\epsilon$ model
How do we compute \( \mu_t \)?

Short answer:

\[
\mu_t = C_\mu \frac{k^2}{\epsilon}
\]

where \( k \) is the turbulence kinetic energy, \( \epsilon \) is the turbulence dissipation rate, and \( C_\mu \) is a constant.

We need to estimate \( k \) and \( \epsilon \)

In the \( k - \epsilon \) model, we solve two transport equations: one for the \( k \) field and one for the \( \epsilon \) field.

Then, the effective viscosity in the Boussinesq model is

\[
\mu_{\text{eff}} = \mu + \mu_t
\]
\[ k - \epsilon \text{ model} \]

Standard equations. See [3].

Transport equation for \( k \)

\[
\frac{Dk}{Dt} = \frac{\partial}{\partial x_i} \left[ \frac{\mu_{\text{eff}}}{\sigma_k} \frac{\partial k}{\partial x_i} \right] + \left[ \mu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho \delta_{ij} k \right] \frac{\partial U_j}{\partial x_i} - c_D \frac{\rho k^{3/2}}{l_m} \\
\text{diffusion} \hspace{1cm} \text{production} \hspace{1cm} \text{dissipation}
\]

Transport equation for \( \epsilon \)

\[
\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_i} \left[ \frac{\mu_{\text{eff}}}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_i} \right] + c_{\epsilon,1} \left[ \mu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho \delta_{ij} k \right] \frac{\partial U_j}{\partial x_i} - c_{\epsilon,1} \frac{\rho \epsilon^2}{k} \\
\text{diffusion} \hspace{1cm} \text{production} \hspace{1cm} \text{dissipation}
\]
What is $\epsilon$?

Use a simple scaling argument to set an upper bound on the dissipation rate (see Panton [1, § 22.10]).

- velocity scale of largest eddy $\sim u_0$
- kinetic energy of the eddy $\sim u_0^2$
- length scale of the eddy $\sim L$
- turn-over time of the eddy $\sim \frac{L}{u_0}$

An upper estimate of the dissipation rate, $\epsilon$ is

$$\epsilon \sim \frac{\text{kinetic energy of an eddy}}{\text{time for one rotation}} = \frac{u_0^2}{L/u_0} = \frac{u_0^3}{L}$$  \hfill (9)
What is $\epsilon$?

But . . . dissipation happens on the smallest scale.

Thus, $\epsilon \sim \frac{u_0^3}{L}$ gives an upper bound, but not a good scale.

At the smallest scales the turbulence is (tends to be) isotropic. The turbulence kinetic energy of isotropic turbulence is

$$k = \frac{3}{2} u' u'$$ isotropic turbulence

Thus, we can estimate the velocity scale as

$$|u'| \sim \sqrt{k}$$ so that $$\epsilon \sim \frac{k^{3/2}}{L}$$
**What is $\epsilon$?**

Introduce a *dissipation length scale*, $\ell_\varepsilon$ such that the dissipation rate is

$$
\epsilon \sim C_D \frac{k^{3/2}}{\ell_\varepsilon}
$$

(10)

where $C_D$ corrects any error in the scale estimate.

But... now we need to estimate $\ell_\varepsilon$

Turn Equation (10) around

$$
\ell_\varepsilon \sim C_D \frac{k^{3/2}}{\epsilon}
$$

(11)

Use Prandtl’s mixing length hypothesis to estimate $\epsilon$ from the mixing length
Prandtl’s mixing length hypothesis

\( \ell_m \) is the length scale for the eddies responsible for mixing, i.e., the most energetic eddies. \( \ell_m > \ell_\varepsilon \).

\[
\ell_m = C' \ell_\varepsilon
\]

Prandtl’s estimate of the turbulence viscosity (eddy viscosity) is

\[
\mu_t \sim \rho V_t \ell_m
\]

where \( V_t \) is the velocity scale

At the dissipation scale, which is isotropic

\[
V_t \sim \sqrt{k}
\]
Prandtl’s mixing length hypothesis

Combine estimates of $V_t, \ell_\varepsilon$ in Prandtl’s expression for eddy viscosity

$$\mu_t = C\rho \left( C_D \frac{k^{3/2}}{\varepsilon} \right) \left( k^{1/2} \right)$$

or

$$\mu_t = C'_\mu \rho \frac{k^2}{\varepsilon}$$  \hspace{1cm} (12)

where $C'_\mu$ is a constant adjusted from experimental values.
Calculation of the eddy viscosity, $\mu_t$, in a CFD code

1. Solve equations for velocities and pressure, using current guess at $\mu_{\text{eff}}$.

2. Solve the $k$ and $\epsilon$ equations

3. At each point in the flow compute

$$\mu_t = C_\mu \rho \frac{k^2}{\epsilon}$$

4. Update $\mu_{\text{eff}} = \mu + \mu_t$.

5. Return to step 1 until convergence.
Wall Functions
Wall Functions

In many engineering flows, we don’t have the computational power to resolve the velocity profile near the wall. One solution is to fudge it with wall functions.
Wall variables

Scaling arguments lead to the inner (near-wall) coordinates

\[ y^+ = \frac{y u_*}{\nu} \]  \hspace{1cm} (13)

\[ u_* = \sqrt{\frac{\tau_w}{\rho}} \hspace{1cm} \text{friction velocity} \] \hspace{1cm} (14)

\[ u^+ = \frac{u}{u_*} \] \hspace{1cm} (15)

In the viscous sublayer

\[ u^+ = y^+ \]

In the log-law (inertial) sublayer

\[ u^+ = c_1 \ln y^+ + c_2 \] \hspace{1cm} (16)
Wall treatment

Wall functions are used to compute $\mu_{eff}$ near solid surfaces.

$k$-$\varepsilon$ model is used to compute $\mu_{eff}$ in the central region of the flow.

Remember the prism layer cells in StarCCM+?
Wall treatment

Adjust the near-wall mesh spacing so that $y^+_I > 30$
References

