ME 352 Supplemental Notes:
Infinite and Truncated Series

1 Learning objectives
After studying these notes you should . . .
- Be able to define an infinite series
- Be able to distinguish geometric series from a power series
- Be able to write the generic formula for a Taylor series
- Be able to write the first three terms of the series representations of $e^x$, $\sin(x)$, and $\cos(x)$.

2 Infinite Series: A Review

2.1 Definitions

Sequence: a function whose domain is a set of positive integers
\[(n, f(n)) : n = 1, 2, 3, \ldots\]

Example: $f(n) = 1/n$

Example: (Fibonacci)
\[
\begin{align*}
  f(1) &= 1 \\
  f(2) &= 1 \\
  f(n) &= f(n-2) + f(n-1) \quad n = 3, 4, \ldots
\end{align*}
\]

Exercise: Write out the first ten terms of the Fibonacci series

2.2 Limit of a Sequence
- $\lim_{n \to \infty} f(n) = L$
- Limit of $f(n)$ exists only if its graph has an asymptote
- Limit may or may not exist

Example:
$f(n) = 1/n$ has the limit 0
\[1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\]
Example:

\( f(n) = \frac{n}{n+1} \) has the limit 1

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2' & 3' & 4' & 5'
\end{array}
\]

2.3 Sequence of Partial Sums

\[ s_1 = u_1 \]
\[ s_2 = u_1 + u_2 \]
\[ s_3 = u_1 + u_2 + u_3 \]
\[
\ldots
\]
\[ s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k \]

Each member of the sequence is a sum of \( n \) terms. The sequence can be defined recursively

\[ s_1 = u_1 \]
\[ s_n = s_{n-1} + u_n \quad n > 1 \]

where \( u_n \) is the \( n \)th term.

2.4 Infinite Series

A series (usually partial sums) with an infinite number of terms

Example:

\[ 1 + 2 + 3 + 4 + \ldots \]

Example:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
2.5 Convergence of Infinite Series

If a series converges, it has a limit. However, the existence of a limit is a necessary condition, not a sufficient condition.

Example

\[ f(n) = 1 + 2 + 3 + 4 + \ldots \ n \] does not converge.

Example

\[ f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \] does not converge.

A series

\[ u_1 + u_2 + \ldots + u_n + \ldots \]

will not converge unless \( \lim_{n \to \infty} u_n = 0 \). This is a necessary, not sufficient condition.
2.6 Geometric Series

An infinite series of the form

\[ a + ax + ax^2 + ax^3 + \ldots + ax^n + \ldots \]

is called a Geometric Series. If \( a \neq 0 \), then the ratio of successive terms is \( x \), i.e.

\[ \frac{ax^n}{a x^{n-1}} = x \]

Consider the sum of the first \( n \) terms of the geometric series

\[ s_n = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} \quad (1) \]

Multiply both sides by \( x \)

\[ xs_n = ax + ax^2 + ax^3 + \ldots + ax^n \quad (2) \]

Subtract Equation (2) from Equation (1)

\[ (1 - x)s_n = a - ax^n = a(1 - x^n) \]

If \( x \neq 1 \) divide both sides by \( 1 - x \) to get

\[ s_n = \frac{a(1 - x^n)}{1 - x} \quad (x \neq 1) \quad (3) \]

Now, assume that \( |x| < 1 \) and take the limit as \( n \to \infty \)

\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(1 - x^n)}{1 - x} = \frac{a}{1 - x} \quad |x| < 1 \]

**Summary** If \( |x| < 1 \), then

\[ a + ax + ax^2 + ax^3 + \ldots + ax^n + \ldots = \frac{a}{1 - x} \]

2.7 Power series expansions

An power series is an expression of the form

\[ \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots \]

The geometric series is a power series with all \( a_k = a \), where \( a \) is a constant. Note that a truncated power series is just a polynomial.
2.8 Taylor series expansions

Taylor series expansions are special power series designed to approximate a function.

Given \( y = f(x) \), we seek polynomials of the form

\[
f_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n
\]

such that \( f_n(x) \) is a good approximation to \( f(x) \).

Consider the constant approximation to \( f(x) \)
\[
f_0(x) = a_0
\]

The best choice of \( a_0 \) is the value of the function at some point. Designate \( \tilde{x} \) as the point where \( f_0(x) \) and \( f(x) \) are supposed to agree. Hence, \( a_0 = f(\tilde{x}) \)

Next consider the linear approximation to \( f(x) \)
\[
f_1(x) = a_0 + a_1x
\]

We want the good agreement at \( \tilde{x} \) so rewrite this as
\[
f_1(x - \tilde{x}) = a_0 + a_1(x - \tilde{x})
\]

The best choice of \( a_0 \) is once again \( f(\tilde{x}) \). Geometric reasoning shows that the best choice of \( a_1 \) is the slope of the function \( f(x) \) at \( x = \tilde{x} \). Therefore, the linear approximation to \( f(x) \) near \( \tilde{x} \) is
\[
f_1(x = \tilde{x}) = f(\tilde{x}) + (x - \tilde{x}) \left. \frac{df}{dx} \right|_{x=\tilde{x}}
\]

Repeating this argument gives the Taylor series with remainder
\[
f(x) = f(\tilde{x}) + (x - \tilde{x}) \left. \frac{df}{dx} \right|_{x=\tilde{x}} + \frac{(x - \tilde{x})^2}{2} \left. \frac{d^2f}{dx^2} \right|_{x=\tilde{x}} + \frac{(x - \tilde{x})^3}{3!} \left. \frac{d^3f}{dx^3} \right|_{x=\tilde{x}} + \ldots + R_n(x, \tilde{x})
\]

2.9 Series Expansions for \( e^x \), \( \sin(x) \) and \( \cos(x) \)

The following series converge for all \(-\infty < x < \infty\). However, it is not practical to evaluate these series for large \( x \)

\[
e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!} + \ldots
\]

\[
\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^kx^{2k}}{(2k)!} + \ldots
\]