1. An $n \times n$ matrix is ill-conditioned if **two or more rows of the matrix are nearly linearly dependent.**.

2. Consider two n element column vectors, u and v, and a small, positive scalar value, δ. A mathematical statement that the two vectors are close enough to be considered equal is $\|u - v\| < \delta$.

3. If A is a 5×5 matrix, x and b are 5×1 (column) vectors, and rank$(A) = 4$, then the solution to $Ax = b$ **may exist, but if it does it will not be unique**.

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4. Given $A^{-1}b = x$ where

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix}, \qquad x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

What is b? Show your work.

Solution:

$$Ax = b \Longrightarrow A^{-1}Ax = A^{-1}b \Longrightarrow Ix = A^{-1}b \Longrightarrow x = A^{-1}b$$

Therefore $x = A^{-1}b \Longrightarrow Ax = b$ (as long as A^{-1} and A exist).

Since A and x are known, $b = Ax$ is obtained by computing the matrix-vector product Ax.

$$Ax = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(2) + (2)(1) \\ (4)(1) + (2)(2) + (4)(1) \\ (2)(1) + (1)(2) + (1)(1) \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 5 \end{bmatrix}.$$

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5. Suppose that a system defined by $Ax = b$ has a numerical solution $\hat{x}$. What is the L_2 norm of the residual for the system of equations if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Show your work.

Solution:

$$r = b - Ax = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} (1)(-1) + (1)(0) + (0)(1) \\ (0)(-1) + (2)(0) + (2)(1) \\ (0)(-1) + (0)(0) + (1)(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$r = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \|r\|_2 = \left[0^2 + 0^2 + 0^2\right]^{1/2} = 0.$$

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6. Given

$$A = \begin{bmatrix} \alpha & 3 & 4 \\ 1 & 0 & \beta \\ 1 & 2 & 3 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 20 \\ 4 \\ 14 \end{bmatrix}$$

what values of α and β are necessary to satisfy $Ax = b$? Show your work.

Solution: Evaluate the inner product between the first row of A and x, and set the result equation to the first element of b.

$$\begin{bmatrix} \alpha & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 20 \Longrightarrow (\alpha)(1) + (3)(2) + (4)(3) = 20 \Longrightarrow \alpha = 20 - 6 - 12 \Longrightarrow \boxed{\alpha = 2}.$$

Repeat for the second row of A

$$\begin{bmatrix} 1 & 0 & \beta \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 4 \Longrightarrow (1)(1) + (0)(2) + (\beta)(3) = 4 \Longrightarrow 3\beta = 3 \Longrightarrow \boxed{\beta = 1}.$$