ME 352 Final Exam Solution Fall 2008

1. Analysis of the effects of roundoff on the numerical solution to a system of n equations in n unknowns leads to the following expression

$$\frac{\|\hat{x} - x\|}{\|\hat{x}\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}$$

where $\hat{x}$ is the numerical solution to $Ax = b$, $\kappa(A)$ is the condition number of A, and $r = b - Ax$. Which of the following statements are true? (Circle the letter for each choice that is true.)

   (b) The equation shows that small residuals are no guarantee that the system was solved accurately.

   The residuals can be small, i.e., $\|r\|/\|b\|$ can be small, but $\kappa(A)$ can be large.

2. Choose the one phrase that best completes the following statement. Given the positive scalar values α and β, the numerical computation of γ with the expression $\gamma = \alpha + \beta$ will have the largest error due to roundoff when

   (a) α is much larger than β.

3. Choose the one phrase that best completes the following statement. Given the positive scalar values α and β, the numerical computation of γ with the expression $\gamma = \alpha - \beta$ will have the largest error due to roundoff when

   (b) α and β are very nearly equal.

   Item a., "α is much larger than β" also causes large roundoff errors, but $\alpha \approx \beta$ causes a catastrophic loss of the most significant digits. Students choosing Item a. were given 2 points out of a possible 5 points.

4. **True or False**: The Matlab backslash operator performs matrix division with a version of Gaussian elimination.

   False. There is no matrix division operation.

5. **True or False**: The Matlab expressions `B = A*A` and `B = A.*A` produce the same matrix `B` only if `A` is a square matrix, i.e., only when `A` has n rows and n columns.

   False: `A*A` and `A.*A` are equal only for exceptional matrices. Consider the arbitrary 2×2 matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

   The normal matrix-matrix product is

$$AA = \begin{bmatrix} a_{1,1}a_{1,1} + a_{1,2}a_{2,1} & a_{1,1}a_{1,2} + a_{1,2}a_{2,2} \\ a_{2,1}a_{1,1} + a_{2,1}a_{2,1} & a_{2,1}a_{1,2} + a_{2,2}a_{2,2} \end{bmatrix}$$

   whereas the element-by-element product (defined only in Matlab) is

$$A.{*}A = \begin{bmatrix} a_{1,1}^2 & a_{1,2}^2 \\ a_{2,1}^2 & a_{2,2}^2 \end{bmatrix}.$$

   Here are just two counter examples using numeric matrices:

```
>> A = [1 2; 3 4];
>> A*A
ans =
     7    10
    15    22
>> A.*A
ans =
     1     4
     9    16

>> B = [1 2; 2 4];
>> B*B
ans =
     5    10
    10    20
>> B.*B
ans =
     1     4
     4    16
```

Note that in the second counter example, `B` is symmetric. Some students claimed that `A*A = A.*A` for symmetric matrices.

6. Choose the one phrase that best completes the following statement. At each iteration of the bisection method, the interval, δ, containing the root

   b. is reduced by a factor of δ.

7. When applied to an $n \times n$ system of equations, the \ operator performs Gaussian elimination with partial pivoting. Consider the following MATLAB session.

```
>> A = rand(5,5);   b = rand(5,1);   x = A\b;   r = b - A*x;
>> fprintf('  %13.3e\n',r)
     4.163e-016
     8.327e-017
     1.887e-015
     1.332e-015
    -9.992e-016
```

   The `fprintf('  %13.3e\n',r)` statement prints the elements of the `r` vector in scientific notation with one element per line. Choose the one phrase that best completes the following statement.

   The values of the elements of `r` indicate

   (c) nothing, the \ operator always gives small residuals.

8. The fourth order Runge-Kutta scheme has a truncation error that is $\mathcal{O}(h^4)$. The midpoint scheme has a truncation error that is $\mathcal{O}(h^2)$. Consider the application of the fourth order Runge-Kutta scheme and the midpoint scheme to the solution of an ODE on an interval $0 \le t \le T_n$. Which of the following statements are true? (Circle all that apply.)

   (e) None of the above.

   This is a hard problem, too hard for this class. I wrote the question on an airplane and debugged it with jet lag. That was a bad idea. I apologize for the stress I caused you in trying to answer this problem on the final exam.

   I was generous in grading by awarding partial credit points to students who attempted to work out the details.

In the classroom demonstration by Bryan, and in several examples in the book, the variation of truncation error with h was demonstrated. The key idea is that the theoretical models for truncation error all prediction of how the discretization error changes when h is changed. These simple expressions (like those given in the problem statement) do not allow us to compare truncation errors for two different methods *at the same* h.

9. Complete the following statement. In contrast to software such as MATLAB that performs numerical calculations, software that performs symbolic calculations. . .

uses exact arithmetic, i.e., all computations are performed with infinite precision.

do not have roundoff, either in storing numbers or in performing calculations with those numbers.

uses algebraic manipulation and simplification to obtain analytical results.

uses direct evaluation of arithmetic expressions only when requested by the user. Typically a numerical result is only obtained after all simplifying calculations have been completed.

Examples of symbolic software: Maple, Mathematica, Derive, Macsyma, . . . See `http://en.wikipedia.org/wiki/Computer_algebra_system` for a list of packages.

Examples of numeric software: MATLAB, MathCAD, Excel, and programming languages such as C, C++, Fortran, Basic, . . .

10. Given matrix A defined to the right.

    (a) Compute A^T

    (b) What is A^{-1}?

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Mea Culpa This problem had a terrible typo. Part (a) should be "Compute $A^T A$". Computation of A^T is so trivial that it should not be worth 5 points. However, given that the typo was my mistake, and that many students appeared to work hard to find A^{-1}, I graded very generously.

On to the solution...

The solution will follow the intended form of the problem, not the problem as written on the exam.

(a) Compute $A^T A$: (note that $A^T = A$, i.e., A is symmetric)

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+1+1+1) & (1-1+1-1) & (1+1-1-1) & (1-1-1+1) \\ (1-1+1-1) & (1+1+1+1) & (1-1-1+1) & (1+1-1-1) \\ (1+1-1-1) & (1+1-1-1) & (1+1+1+1) & (1-1+1-1) \\ (1-1-1+1) & (1+1-1-1) & (1-1+1-1) & (1+1+1+1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, $A^T A = 4I$.

(b) Since $A^T A = 4I$, $(1/4)A^T A = I$ and $(1/4)A^T = A^{-1}$

$$A^{-1} = \frac{1}{4}A^T = \frac{1}{4}A = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

11. The goal is to set up the numerical solution of

$$\frac{dy}{dt} + 4yt = \sin(3\pi t), \qquad y(0) = 1.$$

with the odeRK4 function. Refer to the universal cheat sheet for the listing of odeRK4.

(a) Write the m-file function that defines the ODE in a form suitable for use with odeRK4.

(b) Write the MATLAB statements to call odeRK4 and obtain the solution.

(c) Write the MATLAB statements to plot the $y(t)$ obtained from the odeRK4.

Solution:

(a) Rearrange the ODE into standard form

$$\frac{dy}{dt} + 4yt = \sin(3\pi t) \quad \Longrightarrow \quad \frac{dy}{dt} = \sin(3\pi t) - 4yt$$

The odeFinal.m file contains the code to evaluate the right hand side of the ODE written standard form.

Contents of odeFinal.m:

```
function dydt = odeFinal(t,y)
% odeFinal  Define right hand side of the ODE for problem 11
%           on the ME 352 final exam, Fall 2008
dydt = sin(3*pi*t) - 4*y*t;
```

The correct syntax for inputs and outputs of odeFinal can be deduced from the source code to odeRK4. The m-file must have two inputs t and t, and one output equal to the value of dy/dt at the given values of t and y.

(b) No values for the end-point of the time interval or the step-size (h) are specified in the problem statement. We assume the stopping time is $t_n = 3$ and the stepsize is $h = 0.1$.

```
y0 = 1;
tn = 3;
h = 0.1;
[t,y] = odeRK4('odeFinal',tn,h,y0);  %  or [t,y] = odeRK4(@odeFinal,tn,h,y0);
```

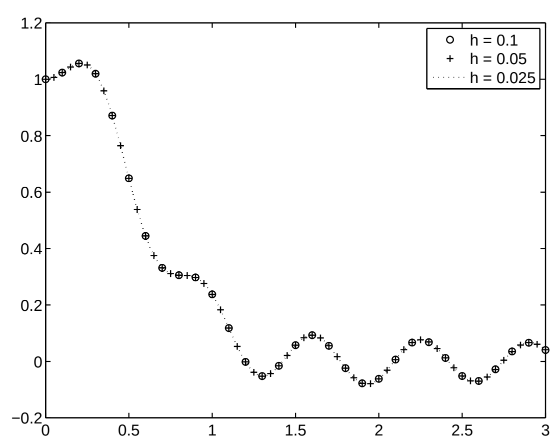
(c) `plot(t,y,'o');`

The code, below left, creates the plot, below right, that shows the numerical solution obtained with odeRK4 and step sizes of $h = 0.1$, $h = 0.05$, and $h = 0.025$.

```
function demoFinalODE
% demoFinalODE  Solve ODE problem 11
%               ME 352 Final Exam, Fall 2008
y0 = 1;
tn = 3;
h = 0.1;
[t,y] = odeRK4(@odeFinal,tn,h,y0);
plot(t,y,'bo');   hold('on')

h = 0.05;
[t,y] = odeRK4(@odeFinal,tn,h,y0);
plot(t,y,'k+');

h = 0.025;
[t,y] = odeRK4(@odeFinal,tn,h,y0);
plot(t,y,'r:');
legend('h = 0.1','h = 0.05','h = 0.025')
```



12. The goal is to find a curve fit of $y = \alpha x + \beta x^2$ to the data in the table to the right.

x	y
-2	-1
0	0.5
1	2

(a) Write the overdetermined system of equations.

(b) Write the normal equations. Show the numerical values, not just the symbols. Show the steps to obtain the normal equations.

(c) When the curve fit coefficients α and β are found, what is the value of the fit function at $x = 0$?

Solution:

(a) Write three equations for the fit function evaluated at each of the three data pairs.

$$\begin{aligned} \alpha x_1 + \beta x_1^2 &= y_1 & \iff && -2\alpha + 4\beta &= -1 \\ \alpha x_2 + \beta x_2^2 &= y_2 & \iff && 0\alpha + 0\beta &= 0.5 \\ \alpha x_3 + \beta x_3^2 &= y_3 & \iff && \alpha + \beta &= 2 \end{aligned}$$

Express the equations in matrix form

$$\begin{bmatrix} x_1 & x_1^2 \\ x_2 & x_2^2 \\ x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \iff \qquad \begin{bmatrix} -2 & 4 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 0.5 \\ 2 \end{bmatrix}$$

The preceding equations can be written symbolically as $Ac = y$ where

$$A = \begin{bmatrix} -2 & 4 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad c = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad y = \begin{bmatrix} -1 \\ 0.5 \\ 2 \end{bmatrix}$$

(b) The normal equations are obtained by multiplying the matrix equation by A^T on the left to obtain $A^T Ac = A^T y$.

$$\begin{aligned} A^T A &= \begin{bmatrix} -2 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-2)(-2) + (0)(0) + (1)(1) & (-2)(4) + (0)(0) + (1)(1) \\ (4)(-2) + (0)(0) + (1)(1) & (4)(4) + (0)(0) + (1)(1) \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -7 & 17 \end{bmatrix} \\ A^T y &= \begin{bmatrix} -2 & 0 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 2 \end{bmatrix} = \begin{bmatrix} (-2)(-1) + (0)(-1) + (1)(2) \\ (4)(-1) + (0)(0.5) + (1)(2) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \end{aligned}$$

Therefore, the numeric form of the normal equations are

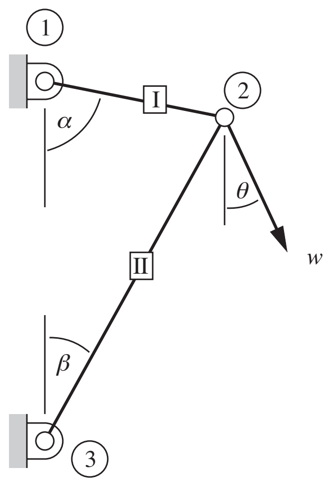
$$\begin{bmatrix} 5 & -7 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

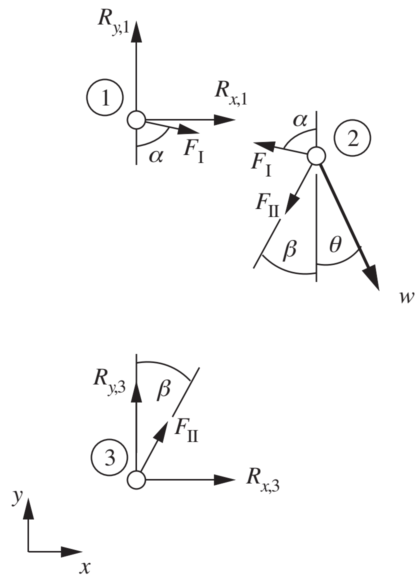
(c) The curve fit function is $\alpha x + \beta x^2 = y$. For any α and β, $x = 0 \Rightarrow y = 0$. There is no need to solve the overdetermined system in order to answer this question.

The solution is $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$. It turns out that the least squares solution is a parabola that passes exactly through the first and third data pairs $(-2, -1)$ and $(1, 2)$. However, the proper *method* of solution is not to simply eliminate the equation for the second data pair $(0, 0.5)$ from the system of equations.

13. The diagram on the left shows a two-link frame attached to a wall by pinned joints. The joints are numbered 1, 2, and 3. The links are labelled I and II.

A force w is applied to joint 2 at an angle θ relative to the vertical. On the right side of diagram, the force components at the three joints are identified. $R_{x,1}$ and $R_{y,1}$ are the reaction forces at joint 1. $R_{x,3}$ and $R_{y,3}$ are the reaction forces at joint 3. F_{I} and F_{II} are the tension forces in links I and II.





The following equations are obtained from force balances on the three joints.

joint 1:
$$R_{x,1} + F_{\mathrm{I}} \sin\alpha = 0$$
$$R_{y,1} - F_{\mathrm{I}} \cos\alpha = 0$$

joint 2:
$$-F_{\mathrm{I}} \sin\alpha - F_{\mathrm{II}} \sin\beta + w \sin\theta = 0$$
$$F_{\mathrm{I}} \cos\alpha - F_{\mathrm{II}} \cos\beta - w \cos\theta = 0$$

joint 3:
$$R_{x,3} + F_{\mathrm{II}} \sin\beta = 0$$
$$R_{y,3} + F_{\mathrm{II}} \cos\beta = 0$$

Assume that α, β, and w are given, and rearrange the preceding equations into a 6×6 system in the form $Ax = b$. The unknowns are the four reaction forces and the two tension forces.

Show the intermediate steps to obtain the matrix form of the equations. Enter your final answer into the form on the next sheet.

There is no special mathematical significance to the dashed lines in the form for recording the matrix equation. The dashed lines are merely used to designate boundaries between elements.

Do not solve the system of equations. Merely set it up.

Solution: As indicated in the problem statement, the unknowns are $R_{x,1}$, $R_{y,1}$, $R_{x,3}$, $R_{y,3}$, F_{I} and F_{II}. Thus, one way to write the x vector is

$$x = \begin{bmatrix} R_{x,1} \\ R_{y,1} \\ R_{x,3} \\ R_{y,3} \\ F_{\mathrm{I}} \\ F_{\mathrm{II}} \end{bmatrix}$$

The order of the unknowns is arbitrary, but once the order of elements in x is chosen, that order must be maintained during the remaining steps of formulating the linear system of equations.

Rewrite the force balance equations to isolate the unknowns. Move all additive constant terms to the right hand side.

$$\begin{aligned} R_{x,1} + (\sin\alpha)F_{\mathrm{I}} &= 0 \\ R_{y,1} - (\cos\alpha)F_{\mathrm{I}} &= 0 \\ -(\sin\alpha)F_{\mathrm{I}} - (\sin\beta)F_{\mathrm{II}} &= -w\sin\theta \\ (\cos\alpha)F_{\mathrm{I}} - (\cos\beta)F_{\mathrm{II}} &= w\cos\theta \\ R_{x,3} + (\sin\beta)F_{\mathrm{II}} &= 0 \\ R_{y,3} + (\cos\beta)F_{\mathrm{II}} &= 0 \end{aligned}$$

The left hand side of each of these equations defines the coefficients of a row in the matrix A. The right hand side of each equation defines the corresponding row in the vector b.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 & -\cos\alpha & 0 \\ 0 & 0 & 0 & 0 & -\sin\alpha & -\sin\beta \\ 0 & 0 & 0 & 0 & \cos\alpha & -\cos\beta \\ 0 & 0 & 1 & 0 & 0 & \sin\beta \\ 0 & 0 & 0 & 1 & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} R_{x,1} \\ R_{y,1} \\ R_{x,3} \\ R_{y,3} \\ F_{\mathrm{I}} \\ F_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

Alternative Formulations:

The equations can be arranged in a different order, or the unknowns can be listed in a different order. In other words, there is more than one valid formulation of the linear system.

If $x = [F_{\mathrm{I}}, F_{\mathrm{II}}, R_{x,1}, R_{y,1}, R_{x,3}, R_{y,3}]^T$, and the equations are left in their original order, the system is

$$\begin{bmatrix} \sin\alpha & 0 & 1 & 0 & 0 & 0 \\ -\cos\alpha & 0 & 0 & 1 & 0 & 0 \\ -\sin\alpha & -\sin\beta & 0 & 0 & 0 & 0 \\ \cos\alpha & -\cos\beta & 0 & 0 & 0 & 0 \\ 0 & \sin\beta & 0 & 0 & 1 & 0 \\ 0 & \cos\beta & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{\mathrm{I}} \\ F_{\mathrm{II}} \\ R_{x,1} \\ R_{y,1} \\ R_{x,3} \\ R_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

If $x = [R_{x,1}, R_{y,1}, F_{\mathrm{I}}, F_{\mathrm{II}}, R_{x,3}, R_{y,3}]^T$, and the equations are left in their original order, the system is

$$\begin{bmatrix} 1 & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & 1 & -\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha & -\sin\beta & 0 & 0 \\ 0 & 0 & \cos\alpha & -\cos\beta & 0 & 0 \\ 0 & 0 & 0 & \sin\beta & 1 & 0 \\ 0 & 0 & 0 & \cos\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,1} \\ R_{y,1} \\ F_{\mathrm{I}} \\ F_{\mathrm{II}} \\ R_{x,3} \\ R_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

If $x = [F_{\mathrm{I}}, F_{\mathrm{II}}, R_{x,1}, R_{y,1}, R_{x,3}, R_{y,3}]^T$, and the equations are left in their original order, the system is

$$\begin{bmatrix} \sin\alpha & 0 & 1 & 0 & 0 & 0 \\ -\cos\alpha & 0 & 0 & 1 & 0 & 0 \\ -\sin\alpha & -\sin\beta & 0 & 0 & 0 & 0 \\ \cos\alpha & -\cos\beta & 0 & 0 & 0 & 0 \\ \sin\beta & 0 & 0 & 0 & 1 & 0 \\ \cos\beta & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{\mathrm{I}} \\ F_{\mathrm{II}} \\ R_{x,1} \\ R_{y,1} \\ R_{x,3} \\ R_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

If $x = [R_{x,1}, F_{\mathrm{I}}, R_{y,1}, F_{\mathrm{II}}, R_{x,3}, R_{y,3}]^T$, and the equations are left in their original order, the system is

$$\begin{bmatrix} 1 & \sin\alpha & 0 & 0 & 0 & 0 \\ 0 & -\cos\alpha & 1 & 0 & 0 & 0 \\ 0 & -\sin\alpha & 0 & -\sin\beta & 0 & 0 \\ 0 & \cos\alpha & 0 & -\cos\beta & 0 & 0 \\ 0 & 0 & 0 & \sin\beta & 1 & 0 \\ 0 & 0 & 0 & \cos\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,1} \\ F_{\mathrm{I}} \\ R_{y,1} \\ F_{\mathrm{II}} \\ R_{x,3} \\ R_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

If $x = [R_{x,1}, R_{y,1}, F_{\mathrm{I}}, F_{\mathrm{II}}, R_{x,3}, R_{y,3}]^T$, and the equations are left in their original order, the system is

$$\begin{bmatrix} 1 & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & 1 & -\cos\alpha & 0 & 0 & 0 \\ 0 & 0 & -\sin\alpha & -\sin\beta & 0 & 0 \\ 0 & 0 & \cos\alpha & -\cos\beta & 0 & 0 \\ 0 & 0 & 0 & \sin\beta & 1 & 0 \\ 0 & 0 & 0 & \cos\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{x,1} \\ R_{y,1} \\ F_{\mathrm{I}} \\ F_{\mathrm{II}} \\ R_{x,3} \\ R_{y,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \\ 0 \\ 0 \end{bmatrix}$$

If the equations are listed in this order

$$\begin{aligned} R_{x,1} + (\sin\alpha)F_{\mathrm{I}} &= 0 \\ R_{y,1} - (\cos\alpha)F_{\mathrm{I}} &= 0 \\ R_{x,3} + (\sin\beta)F_{\mathrm{II}} &= 0 \\ R_{y,3} + (\cos\beta)F_{\mathrm{II}} &= 0 \\ -(\sin\alpha)F_{\mathrm{I}} - (\sin\beta)F_{\mathrm{II}} &= -w\sin\theta \\ (\cos\alpha)F_{\mathrm{I}} - (\cos\beta)F_{\mathrm{II}} &= -w\cos\theta \end{aligned}$$

and the unknown vector is chosen to be $x = [R_{x,1}, R_{y,1}, R_{x,3}, R_{y,3}, F_{\mathrm{I}}, F_{\mathrm{II}}]^T$, the system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 & -\cos\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & \sin\beta \\ 0 & 0 & 0 & 1 & 0 & \cos\beta \\ 0 & 0 & 0 & 0 & -\sin\alpha & -\sin\beta \\ 0 & 0 & 0 & 0 & \cos\alpha & -\cos\beta \end{bmatrix} \begin{bmatrix} R_{x,1} \\ R_{y,1} \\ R_{x,3} \\ R_{y,3} \\ F_{\mathrm{I}} \\ F_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -w\sin\theta \\ w\cos\theta \end{bmatrix}$$

There are other valid arrangements of the system.

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