Introduction to Compressible Flow

ME 322 Lecture Slides, Winter 2007

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Overview

Topics

- Basic Concepts
- Review of bulk compressibility in liquids and gases
- Ideal Gas Relationships
- Speed of sound
- Mach number
- When are compressible effects important?
- Isentropic, compressible flow in ducts with variable area

Overview

Learning Objectives

- Be able to list fluid properties associated with compressible flow.
- Be able to use and manipulate isentropic relationships between p, T, and ρ, e.g., $p/\rho^k =$ constant
- Be able to write (from memory) and correctly use the formula for speed of sound of an ideal gas.
- Be able to compute the Mach number and use its value to correctly identify the flow regime.
- Be able to predict whether a compressible flow will increase or decrease as a result of area change and the current value of Ma.
- Be able to evaluate the isentropic relationships for the stagnation properties
- Be able to explain the physical significance of the * states.

Basic Concepts

Incompressible:

Density variations are not important in determining the dynamics of the fluid motion. Small changes in density do not affect velocity and pressure.

Equations governing fluid motion are

- Mass conservation (continuity)
- Momentum conservation
- Energy equation *only if* fluid experiences heat and work interactions

Compressible:

Density variations are important in determining the dynamics of the fluid motion. Changes in density do affect velocity and pressure.

Equations governing fluid motion are

- Mass conservation (continuity)
- Momentum conservation
- Energy conservation
- Equation of state

Applications where Compressible Flow is Important

- High speed aeronautics: jet airplanes, rockets, ballistics
- Gas turbines and compressors, vapor power cycles
- Gas transmission lines (factories, natural gas supply)
- Acoustics: audio equipment, phase change ink jet printers, noise abatement
- Free convection
- Water hammer

Bulk Compressibility in Liquids and Gases (1)

Bulk modulus (MYO, §1.7.1, pp. 20–21)

$$E_v = -\frac{dp}{d\mathcal{V}/\mathcal{V}} = \mathcal{V}\frac{dp}{d\mathcal{V}} \qquad (\star)$$

where $\mathcal{V}$ is the volume of the liquid.

An equivalent formula for E_v is

$$E_v = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho} \qquad (\star\star)$$

where ρ is the volume of the liquid.

Note that Equation $(\star)$ and Equation $(\star\star)$ have different signs on the right hand sides.

Bulk Compressibility in Liquids and Gases (2)

Isothermal compressibility

$$\alpha = -\frac{1}{\hat{v}}\left(\frac{\partial \hat{v}}{\partial p}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$$

where $\hat{v}$ is the specific volume of the fluid.

Volumetric Thermal Expansion coefficient — important in free convection problems

$$\beta = \frac{1}{\hat{v}}\left(\frac{\partial \hat{v}}{\partial T}\right)_p = -\rho\left(\frac{\partial \rho}{\partial p}\right)_p$$

See Çengel and Boles, Chapter 11 for a discussion of these compressibility properties. (p. 617 in fourth edition)

Bulk Compressibility in Liquids and Gases (3)

Now what?

- Bulk compressibility is a material property
- *Key Question*: How does compressibility affect fluid motion?
- Before studying the equations of motion, we'll review ideal gas relationships
- Compressible flow is complex, we will only be introducing the simplest models that are most likely to be of use to the broadest population of practicing engineers.

Review of Ideal Gas Relationships (1)

Ideal Gas Equation

$$p = \rho R T \qquad p = \frac{m}{\mathcal{V}} R T$$

where

ρ	is the gas density,
p	is the *absolute pressure*,
T	is the *absolute temperature*,
$R = \dfrac{\mathcal{R}_u}{\mathcal{M}}$	is the gas constant,
$\mathcal{R}_u$	is the universal gas constant, $\mathcal{R}_u = 8315 \, \frac{\mathrm{J}}{\mathrm{kg \cdot mol \, K}} = 49709 \, \frac{\mathrm{ft \cdot lb_f}}{\mathrm{slug \cdot mol \, {}^\circ R}} = 1545 \, \frac{\mathrm{ft \cdot lb_f}}{\mathrm{lb_m \cdot mol {}^\circ R}}$
$\mathcal{M}$	is the molecular weight of the gas. $\mathcal{M}_{\mathrm{air}} = 28.97 \, \frac{\mathrm{lbm}}{\mathrm{lb_m \cdot mol}} = 28.97 \, \frac{\mathrm{slug}}{\mathrm{slug \cdot mol}} = 28.97 \, \frac{\mathrm{kg}}{\mathrm{kg \cdot mol}}$

Review of Ideal Gas Relationships (1)

Specific Heats

$$c_v \equiv \left(\frac{\partial \breve{u}}{\partial T}\right)_v \qquad c_p \equiv \left(\frac{\partial \breve{h}}{\partial T}\right)_p$$

$\breve{u}$ is the *specific internal energy*

$\breve{h}$ is the *specific enthalpy*

Review of Ideal Gas Relationships (2)

Specific Internal Energy

For an ideal gas $\breve{u}$ is a function of temperature only

$$\breve{u} = \breve{u}(T) \quad \Longrightarrow \quad c_v \equiv \left(\frac{\partial \breve{u}}{\partial T}\right)_v = \frac{d\breve{u}}{dT} \quad \Longrightarrow \quad d\breve{u} = c_v dT$$

Therefore

$$\breve{u}_2 - \breve{u}_1 = \int_{T_1}^{T_2} c_v dT$$

Often we assume that c_v is constant so that the integral reduces to

$$\breve{u}_2 - \breve{u}_1 = \bar{c_v}(T_2 - T_1)$$

Where $\bar{c_v}$ is the *average* value of c_v for the temperature range of interest.

Review of Ideal Gas Relationships (3)

Specific Enthalpy

$$\check{h} = \check{u} + \frac{p}{\rho}$$

so if $\check{u} = \check{u}(T)$, then $\check{h} = \check{h}(T)$

$$\check{h} = \check{h}(T) \quad \Longrightarrow \quad c_p \equiv \left(\frac{\partial \check{h}}{\partial T}\right)_p = \frac{d\check{h}}{dT} \quad \Longrightarrow \quad d\check{h} = c_p dT$$

Therefore

$$\check{h}_2 - \check{h}_1 = \int_{T_1}^{T_2} c_p dT$$

Often we assume that c_p is constant so that the integral reduces to

$$\check{h}_2 - \check{h}_1 = \bar{c_p}(T_2 - T_1)$$

Where $\bar{c_p}$ is the *average* value of c_p for the temperature range of interest.

Review of Ideal Gas Relationships (4)

Specific Heat Relationships for Ideal Gases

$$\left.\begin{aligned} p &= \rho R T \\ \check{h} &= \check{u} + \frac{p}{\rho} \end{aligned}\right\} \implies \check{h} = \check{u} + RT$$

Differentiate the preceding relationship with respect to T

$$\frac{d\check{h}}{dT} = \frac{d\check{u}}{dT} + R \implies c_p = c_v + R \implies \boxed{c_p - c_v = R}$$

Define $k \equiv \dfrac{c_p}{c_v}$ so that

$$\boxed{c_p = \frac{Rk}{k-1}} \quad \text{and} \quad \boxed{c_v = \frac{R}{k-1}}$$

Review of Ideal Gas Relationships (5)

Entropy Relationships

Without approximation the following relationships hold for a pure (single component) substance

$$ds = \frac{c_v}{T} dT + \left(\frac{\partial p}{\partial T}\right)_v dv \qquad\qquad ds = \frac{c_p}{T} dT - \left(\frac{\partial v}{\partial T}\right)_p dp$$

See Çengel and Boles, Chapter 11 (pp. 615–616 in fourth edition)

For an *ideal gas* with *constant specific heats* the preceding equations can be integrated directly to give

$$\boxed{s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{\rho_1}{\rho_2}} \qquad\qquad \boxed{s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}}$$

Review of Ideal Gas Relationships (6)

Isentropic Relationships

State changes that are *reversible and adiabatic* are also *isentropic*.

The first Δs relationship gives

$$\Delta s = 0 \quad \Longrightarrow \quad c_v \ln \frac{T_2}{T_1} = -R \ln \frac{\rho_1}{\rho_2} \quad \Longrightarrow \quad \ln \left(\frac{T_2}{T_1}\right)^{c_v} = \ln \left(\frac{\rho_2}{\rho_1}\right)^{R}$$

$$\left(\frac{T_2}{T_1}\right)^{c_v} = \left(\frac{\rho_2}{\rho_1}\right)^{R}$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{R/c_v}$$

$$\therefore \quad \boxed{\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k-1}}$$

Review of Ideal Gas Relationships (7)

Isentropic Relationships

The second Δs relationship gives

$$\Delta s = 0 \quad \Longrightarrow \quad c_p \ln \frac{T_2}{T_1} = R \ln \frac{p_2}{p_1} \quad \Longrightarrow \quad \ln\left(\frac{T_2}{T_1}\right)^{cp} = \ln\left(\frac{p_2}{p_1}\right)^{R}$$

$$\left(\frac{T_2}{T_1}\right)^{cp} = \left(\frac{p_2}{p_1}\right)^{R}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{R/cp}$$

$$\therefore \quad \boxed{\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{(k-1)/k}}$$

Review of Ideal Gas Relationships (8)

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k-1} \quad \text{and} \quad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{(k-1)/k} \qquad \Longrightarrow \qquad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{k}$$

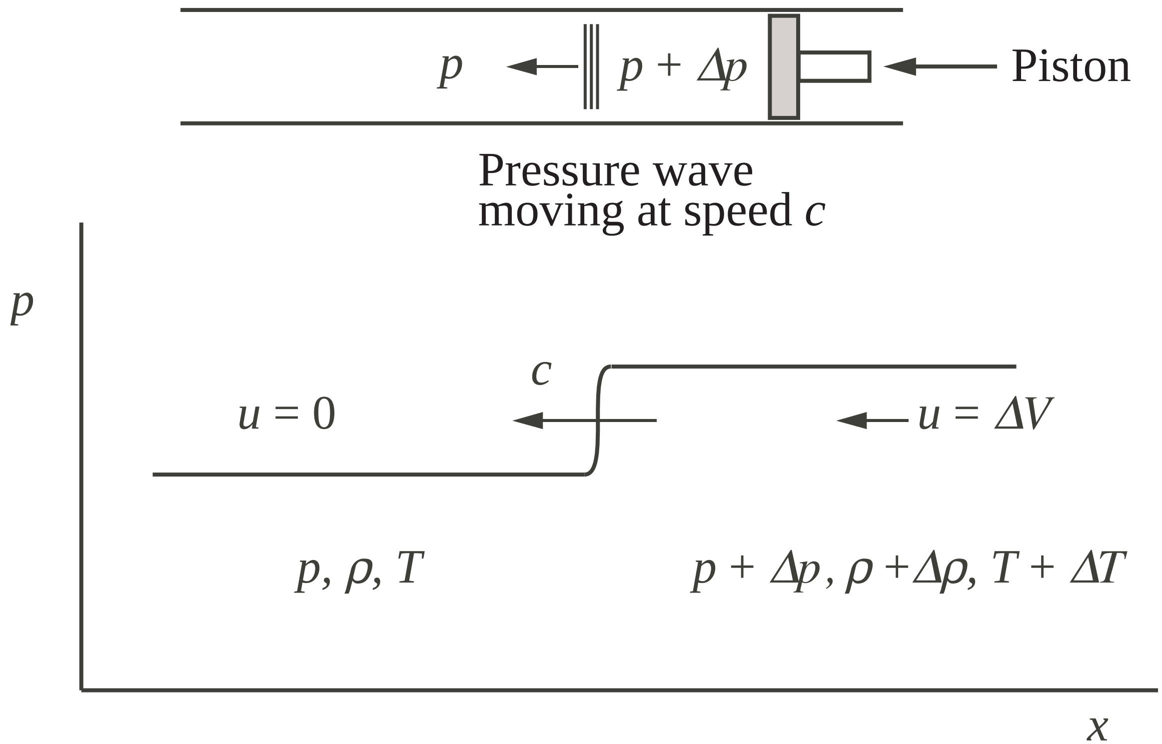
Speed of Sound (1)

Overview

- Thought experiment of a pressure pulse traveling in a tube
- Apply momentum and mass conservation for a moving control volume
- Use ideal gas relationships
- Result: $c = \sqrt{kRT}$

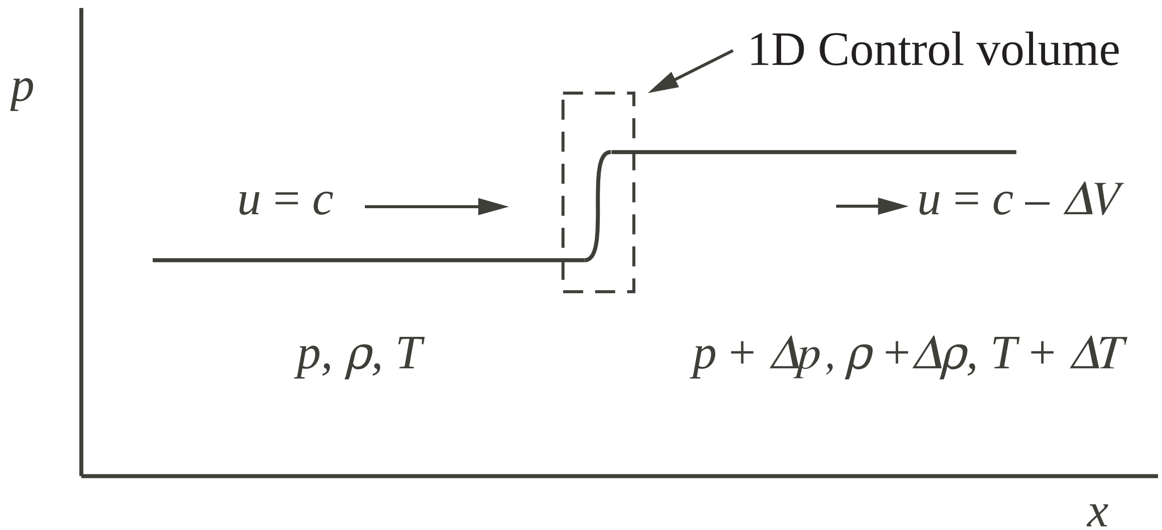
Speed of Sound (2)

Consider a pressure pulse moving through a tube



Speed of Sound (3)

Change frame of reference: Move with the pulse



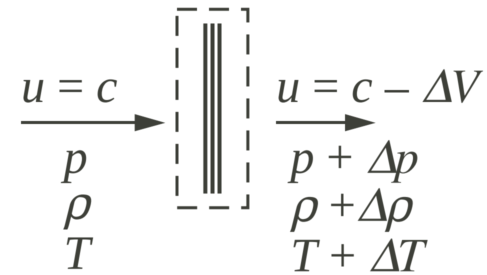
Speed of Sound (4)

Apply mass conservation:

$$\rho A c = (\rho + \Delta\rho) A (c - \Delta V)$$

$$\Longrightarrow \qquad \Delta V = c \frac{\Delta\rho}{\rho + \Delta\rho} \qquad (1)$$

$$(2)$$

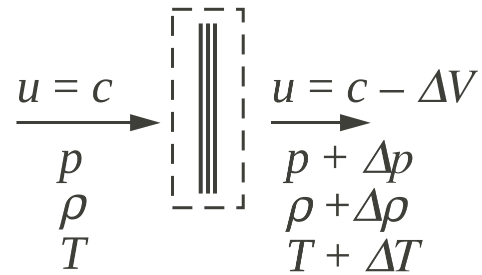


We assume that the pulse is a small pressure perturbation. This is consistent with observations of sound waves.

$$\frac{\Delta p}{p} \ll 1 \quad \Longrightarrow \quad \frac{\Delta\rho}{\rho} \ll 1 \quad \Longrightarrow \quad \Delta V \ll c$$

Speed of Sound (5)

Apply momentum conservation:



$$\sum F_x = \dot{m}\big(V_{\text{out}} - V_{\text{in}}\big)$$

$$\Longrightarrow pA - \big(p + \Delta p\big)A = \rho Ac\Big[\big(c + \Delta V\big) - c\Big]$$

which simplifies to

$$\Delta V = \frac{\Delta p}{\rho c} \tag{3}$$

Speed of Sound (6)

Combining Equation (1) and Equation (3) to eliminate ΔV gives

$$\frac{\Delta p}{\rho c} = c \frac{\Delta \rho}{\rho + \Delta \rho}$$

or

$$c^2 = \frac{\Delta p}{\Delta \rho}\left(1 + \frac{\Delta \rho}{\rho}\right) \tag{4}$$

Recall that c is the speed of the disturbance propagating through the fluid.

For small pressure disturbances $\Delta \rho / \rho \ll 1$ and Equation (4) reduces to

$$c^2 = \frac{\Delta p}{\Delta \rho} \tag{5}$$

Speed of Sound (7)

Assume:

- The disturbance is small, i.e. $\Delta p/p \ll 1$, $\Delta \rho/\rho \ll 1$
- Friction and heat transfer are negligible *in the control volume*
  $\Longrightarrow$ The passing pulse is *adiabatic*, *reversible*, and therefore *isentropic*

Under these assumptions Equation (5) is equivalent to

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \qquad \text{or} \qquad c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{6}$$

Thus c is a *thermodynamic* property of the substance.

c is called the *speed of sound* because sound transmission occurs via small pressure perturbations consistent with the assumptions used in the derivation.

Speed of Sound (8)

Sound Speed for Ideal Gases

For an isentropic process of an ideal gas

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^k \quad \Longrightarrow \quad p_2 = p_1 \left(\frac{\rho_2}{\rho_1}\right)^k = \frac{p_1}{\rho_1^k}\,\rho_2^k \quad \Longrightarrow \quad p = B\rho^k$$

where $B = p/\rho^k$ is a constant.

$$\Longrightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = Bk\rho^{k-1} = \left(\frac{p}{\rho^k}\right)k\rho^{k-1} = \frac{p}{\rho}\,k$$

From the ideal gas equation $\dfrac{p}{\rho} = RT$, so, for an ideal gas

$$\left(\frac{\partial p}{\partial \rho}\right)_s = kRT \qquad \text{and} \qquad \boxed{c = \sqrt{kRT}}$$

Speed of Sound (9)

In General

The definition of bulk modulus is

$$E_v = \rho \left(\frac{\partial p}{\partial \rho} \right)_s \qquad \Longrightarrow \qquad \left(\frac{\partial p}{\partial \rho} \right)_s = \frac{E_v}{\rho}$$

Therefore

$$c = \sqrt{\frac{E_v}{\rho}}$$

which applies to liquids and solids, as well as gases.

Mach Number

The Mach number is named after Ernst Mach (1838–1916), a physicist and philosopher who made early contributions to our understanding of compressible flow.

Definition:

$$\mathrm{Ma} = \frac{V}{c} = \frac{V}{\sqrt{kRT}}$$

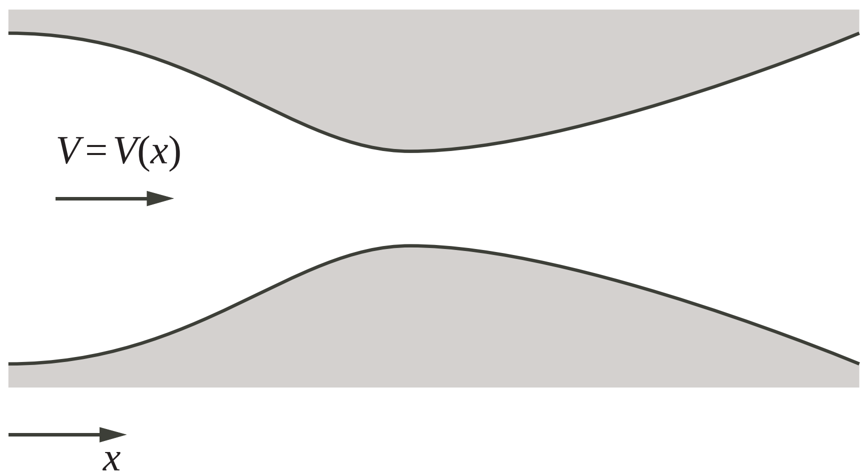
Ma depends on the local velocity V, and varies throughout a compressible flow.

Ma Range	Flow Nomenclature and Features
$0 \leq \mathrm{Ma} < 0.3$	Incompressible flow, density variations are neglected
$\mathrm{Ma} < 1$	Subsonic flow
$\mathrm{Ma} \sim 1$	Transonic flow
$\mathrm{Ma} = 1$	Sonic flow
$\mathrm{Ma} > 1$	Supersonic flow
$\mathrm{Ma} > 5$	Hypersonic flow

Ducts with Area Change (1)

One-Dimensional, Isentropic Flow Model

A useful engineering model can be obtained by neglecting heat transfer, friction, and other irreversible effects in short ducts. We also need to neglect boundary layer effects and assume the variations in the x direction (flow direction) are dominant.



Ducts with Area Change (2)

Mass Conservation

$$\dot{m} = \rho V A = \text{constant} \qquad \Longrightarrow \qquad \frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

Momentum Conservation (no viscous shear, no external work interactions)

$$dp + \frac{1}{2}\rho d(V^2) + \rho g dz = 0 \qquad \text{neglect } dz \text{ for gases} \qquad \Longrightarrow \qquad \frac{dp}{\rho V^2} = -\frac{dV}{V}$$

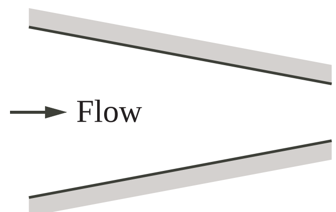
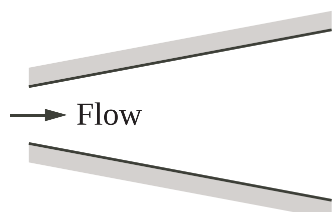
Combine Mass and Momentum Equations

$$\frac{dV}{V} = \frac{-1}{1 - \mathrm{Ma}^2}\frac{dA}{A} \qquad\qquad \frac{dp}{p} = \frac{\mathrm{Ma}^2}{1 - \mathrm{Ma}^2}\frac{dA}{A}$$

These equations give us some insight into the strangeness of supersonic flow.

Ducts with Area Change (3)

$$\frac{dV}{V} = \frac{-1}{1 - \mathrm{Ma}^2}\frac{dA}{A}$$

	Subsonic flow $\mathrm{Ma} < 1$	Supersonic flow $\mathrm{Ma} > 1$
	$dA < 0$ $dV > 0$	$dA < 0$ $dV < 0$
	$dA > 0$ $dV < 0$	$dA > 0$ $dV > 0$

Isentropic Flow in Ducts with Area Change (1)

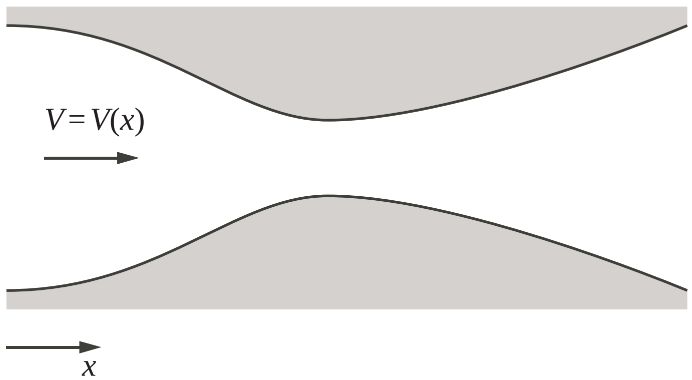
Combine mass and momentum conservation with isentropic relations for ideal gases to get

$$c_p(T_0 - T) - \frac{V^2}{2} = 0$$

or

$$T_0 = T + \frac{V^2}{2c_p}$$

where T_0 is the *stagnation temperature* along the flow path.



Stagnation states are reference conditions that are constant along an isentropic flow.

Isentropic Flow in Ducts with Area Change (2)

$$\boxed{\frac{T}{T_0} = \frac{1}{1 + \frac{k-1}{2}\mathrm{Ma}^2}}$$

$$\boxed{\frac{p}{p_0} = \left[\frac{1}{1 + \frac{k-1}{2}\mathrm{Ma}^2}\right]^{k/(k-1)}}$$

$$\boxed{\frac{\rho}{\rho_0} = \left[\frac{1}{1 + \frac{k-1}{2}\mathrm{Ma}^2}\right]^{1/(k-1)}}$$

Given Ma it is straightforward to find T/T_0, but given T/T_0 an iterative root-finding procedure is necessary to compute Ma. These equations are tabulated so that it's possible to work them in both directions. See Munson, Young, and Okiishi, Figure D.1, p. 7.66

Isentropic Flow in Ducts with Area Change (3)

Since T varies along the flow, so does the speed of sound.

Define:

$$c = \sqrt{kRT} \quad \text{is the sound speed at temperature } T$$

$$c_0 = \sqrt{kRT_0} \quad \text{is the sound speed at temperature } T_0$$

This leads to

$$\frac{c}{c_0} = \left[\frac{T}{T_0}\right]^{1/2} = \left[\frac{1}{1 + \dfrac{k-1}{2}\,\mathrm{Ma}^2}\right]^{1/2}$$

Isentropic Flow in Ducts with Area Change (4)

Isentropic flow relationships at sonic conditions

Use $*$ to designate conditions at $\mathrm{Ma} = 1$.

$$\frac{T^*}{T_0} = \frac{2}{k+1} \qquad \frac{T^*}{T_0} = 0.8333 \quad \text{for } k = 1.4$$

$$\frac{p^*}{p_0} = \left[\frac{2}{k+1}\right]^{k/(k-1)} \qquad \frac{p^*}{p_0} = 0.5283 \quad \text{for } k = 1.4$$

$$\frac{\rho^*}{\rho_0} = \left[\frac{2}{k+1}\right]^{1/(k-1)} \qquad \frac{\rho^*}{\rho_0} = 0.6339 \quad \text{for } k = 1.4$$

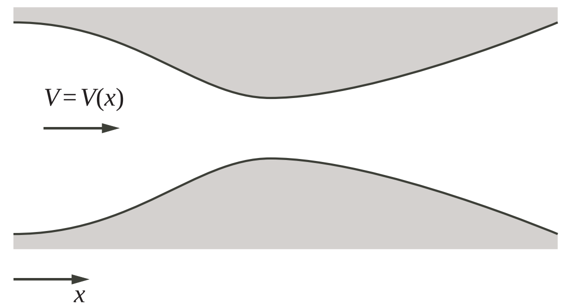
$$\frac{c^*}{c_0} = \left[\frac{2}{k+1}\right]^{1/2} \qquad \frac{c^*}{c_0} = 0.9129 \quad \text{for } k = 1.4$$

Isentropic Flow in Ducts with Area Change (5)

Flow Rate Calculations

The mass flow rate at any point along the duct is

$$\dot{m} = \rho A V = \rho^* A^* V^*$$



The properties ρ^*, A^*, V^* exist at the cross section where $\mathrm{Ma} = 1$.

The * states are reference properties *even if* the flow does not have $\mathrm{Ma} = 1$ anywhere.

Isentropic Flow in Ducts with Area Change (6)

Choked Flow

- If flow is sonic in the duct, $\mathrm{Ma} = 1$ at the minimum area
- If $\mathrm{Ma} = 1$ at the minimum area the flow is *choked*
- For choked flow, reducing the downstream pressure cannot increase the flow rate.

The choked flow state is

$$\dot{m} = \rho^* V^* A^*$$

Note that we can identify (and compute) ρ^*, V^*, and A^* even if the flow is not choked anywhere in the duct.

Isentropic Flow in Ducts with Area Change (7)

Since the flow steady, mass conservation requires

$$\rho A V = \rho^* A^* V^* \Longrightarrow \qquad \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

Algebraic manipulations yield

$$\frac{V^*}{V} = \frac{\sqrt{kRT^*}}{\mathrm{Ma}\sqrt{kRT}} \qquad \frac{\rho^*}{\rho} = \left[\frac{2}{k+1}\left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{1/(k-1)}$$

$$\frac{T^*}{T} = \left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)$$

$$\boxed{\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\frac{2}{k+1}\left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{(k+1)/[2(k-1)]}}$$

Isentropic Flow in Ducts with Area Change (8)

The *maximum* possible mass flow rate through a duct is

$$\dot{m}_{\max} = \dot{m}^* = \rho^* A^* V^*$$

Substitute

$$\rho^* = \rho_0 \left(\frac{2}{k+1}\right)^{1/(k-1)}, \qquad V^* = \sqrt{kRT^*} = \left(\frac{2k}{k+1} RT_0\right)^{1/2}$$

and simplify to get

$$\dot{m}_{\max} = \left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]} A^* \rho_0 \sqrt{kRT_0}$$

Isentropic Flow in Ducts with Area Change (9)

For air ($k = 1.4$)

$$\dot{m}_{\mathrm{max,air}} = \frac{0.6847 p_0 A^*}{\sqrt{RT_0}}$$

Note: $\dot{m}_{\mathrm{max}}$ is unaffected by downstream conditions!

That is the nature of choked flow.