Boundary Layer Analysis

ME 322 Lecture Slides, Winter 2007

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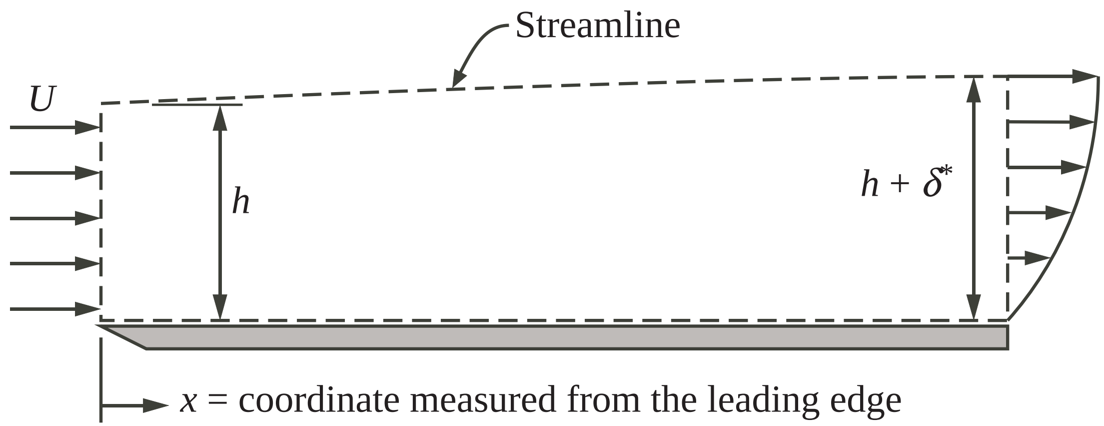
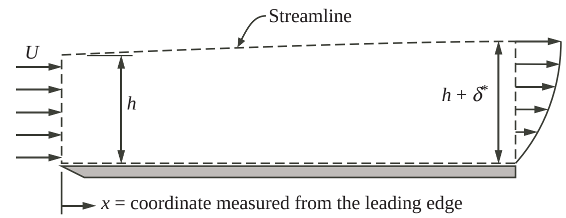
Displacement Thickness (1)



δ^* is the amount by which the streamline just outside the boundary layer is displaced.

Displacement Thickness (2)

Apply mass conservation to the control volume



$$\int_{CS} \rho(\vec{V} \cdot \hat{\boldsymbol{n}})dA = 0$$

$$-\int_0^h \rho U b dy + \int_0^{h+\delta^*} \rho u(y) b dy = 0$$

$$-\rho U b h + \int_0^{h+\delta^*} \rho u(y) b dy = 0$$

$$\Longrightarrow \quad Uh = \int_0^{h+\delta^*} u(y) dy \qquad (\star)$$

Displacement Thickness (3)

Continue . . . add and subtract U to the integrand on the right hand side of Equation ($\star$).

$$Uh = \int_0^{h+\delta^*} (U - U + u(y))dy \;=\; U(h+\delta^*) + \int_0^{h+\delta^*} (u(y) - U)dy$$

Solve for δ^*

$$\delta^* = \frac{1}{U}\int_0^{h+\delta^*} (U - u(y))dy \;=\; \int_0^{h+\delta^*} \left(1 - \frac{u(y)}{U}\right)dy$$

Displacement Thickness (4)

The preceding analysis shows tht

$$\delta^* = \int_0^{h+\delta^*} \left(1 - \frac{u(y)}{U}\right) dy$$

Since $u(y) = U =$ constant outside the boundary layer, the upper limit is arbitrary as long as h and $h + \delta^*$ are outside the boundary layer. So, we can change the upper limit of integration to ∞

$$\boxed{\delta^* = \int_0^{\infty} \left(1 - \frac{u(y)}{U}\right) dy}$$

Scale Analysis for Laminar Boundary Layers (1)

Assume the boundary layer is thin, i.e. assume $\dfrac{\delta}{L} \ll 1$

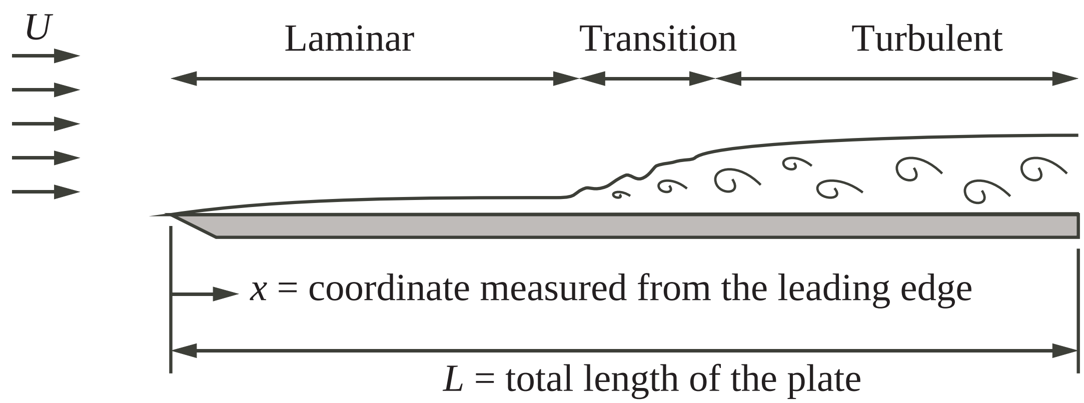
The continuity equation requires that v is small, i.e. $v \sim U\dfrac{\delta}{L}$

The x direction momentum equation requires that $\dfrac{\delta}{L} \sim \mathrm{Re}_L^{-1/2}$

Therefore $\dfrac{\delta}{L}$ will be small *if* Re_L is large.

Generally we take $\mathrm{Re}_L \approx 1000$ as the *minimum* Re_L for a boundary layer to exist.

Boundary Layer Flow Regimes

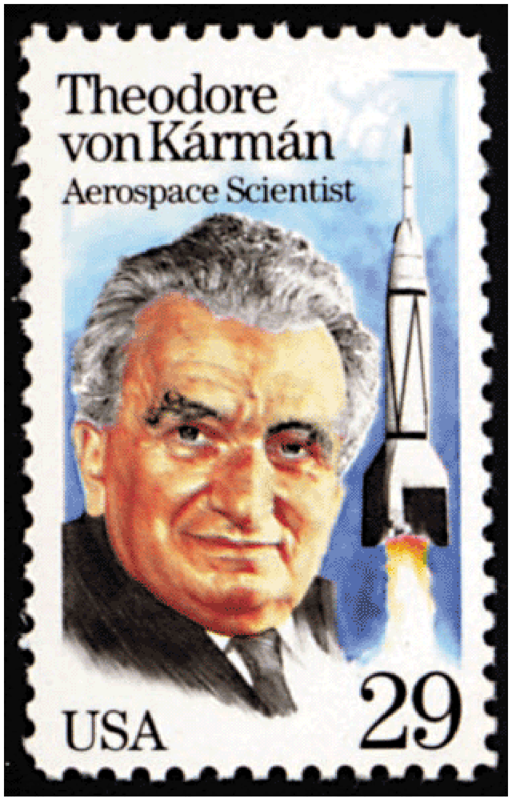


$$\mathrm{Re}_x = \frac{\rho U x}{\mu} \qquad \mathrm{Re}_L = \frac{\rho U L}{\mu}$$

The critical Reynolds number for transition from laminar to turbulent flow is

$$\mathrm{Re}_{cr} \approx 5 \times 10^5$$

Integral Analysis for Laminar Boundary Layers (1)



http://en.wikipedia.org/wiki/Theodore_Von_Karman

Integral Analysis for Laminar Boundary Layers (2)

Derive momentum integral for flat plate — MYO, Equation (9.22), p 502.

$$D(x) = \rho b \int_0^{\delta(x)} u(U-u)dy \tag{1}$$

von Kàrmàn wrote equation (1) as

$$D(x) = \rho b U^2 \theta \tag{2}$$

where

$$\theta = \int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right)\, dy \tag{3}$$

is called the *momentum thickness*.

θ is a measure of total plate drag. Note that θ has dimensions of length.

Integral Analysis for Laminar Boundary Layers (3)

Since the plate is parallel to the on-coming flow, the drag is only due to wall shear stress

$$D(x) = b \int_0^x \tau_w(x)\, dx \tag{4}$$

Take derivatives of equation (4) and (2)

$$\frac{dD}{dx} = b\tau_w \tag{5}$$

Integral Analysis for Laminar Boundary Layers (4)

Assume U is constant and take derivative of equation (2)

$$\frac{dD}{dx} = \rho b U^2 \frac{d\theta}{dx} \qquad (6)$$

Combine equation (5) and equation (6)

$$\boxed{\tau_w = \rho U^2 \frac{d\theta}{dx}} \qquad \begin{array}{l}\text{constant } U \\ \text{laminar or turbulent}\end{array} \qquad (7)$$

Integral Analysis for Laminar Boundary Layers (5)

Summary so far. We have the von Kàrmàn integral momentum equation

$$\boxed{\tau_w = \rho U^2 \frac{d\theta}{dx}} \tag{7}$$

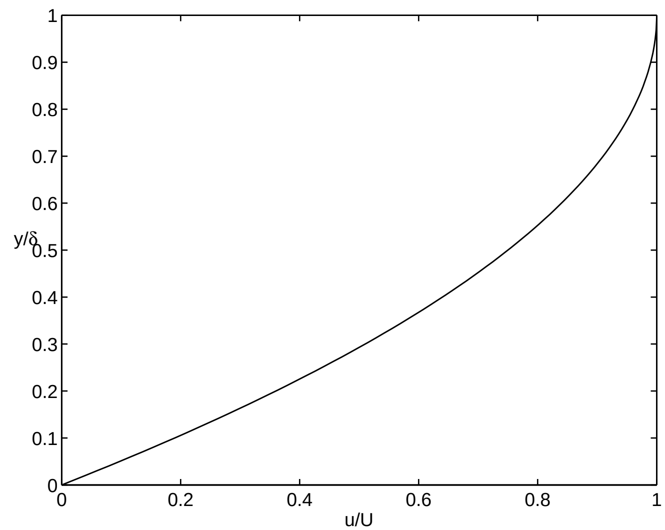
- Equation (7) relates the *local* wall shear stress to the *local* momentum thickness. Both τ_w and θ vary with position along the plate.
- Equation (7) is a tool for analysis of flat plate boundary layers. All we need to do is make assumptions for the profile *shape*, i.e., $\dfrac{u}{U} = \mathrm{fcn}\left(\dfrac{y}{\delta}\right)$, and equation (7) will allow us to calculate $\tau_w(x)$, and from there, $D(x)$ and D_{total}

Integral Analysis for Laminar Boundary Layers (6)

Apply von Kàrmàn parabolic profile:

Assume

$$\frac{u}{U} = 2\frac{y}{\delta} - \frac{y^2}{\delta^2}$$



Substitute into definition of θ

$$\theta = \int_0^\delta \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)\,dy \qquad = \frac{2}{15}\delta$$

Integral Analysis for Laminar Boundary Layers (7)

Substitute parabolic profile into definition of τ_w

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{\partial u}{\partial y} = U\left(\frac{2}{\delta} - 2\frac{y}{\delta}\right) \qquad \Longrightarrow \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{2}{\delta}$$

$$\therefore \tau_w = \frac{2\mu U}{\delta}$$

Integral Analysis for Laminar Boundary Layers (8)

Put the pieces back into equation (7)

$$\tau_w = \rho U^2 \frac{d\theta}{dx} \qquad \Longrightarrow \qquad \frac{2\mu U}{\delta} = \rho U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

Rearrange

$$\delta d\delta = 15\,\frac{\nu}{U}\,dx$$

where $\nu = \mu/\rho$

Integrate to get

$$\frac{1}{2}\delta^2 = \frac{15\nu x}{U}$$

or

$$\frac{\delta}{x} = 5.5\left(\frac{\nu}{Ux}\right)^{1/2} = 5.5\mathrm{Re}_x^{-1/2}$$

Integral Analysis for Laminar Boundary Layers (9)

Now we know $u/U = \mathrm{fcn}(y/\delta)$. From this velocity profile we can compute the wall shear stress

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{2\mu U}{\delta} = (2\mu U)\ (5.5x\mathrm{Re}_x^{-1/2})$$

Make dimensionless as c_f

$$c_f = \frac{\tau_2}{(1/2)\rho U^2} = \sqrt{\frac{8}{15}}\mathrm{Re}_x^{-1/2} = \frac{0.73}{\mathrm{Re}_x^{1/2}}$$

Integral Analysis for Laminar Boundary Layers (10)

Recall definition of displacement thickness

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right)\, dy$$

So the displacement thickness for parabolic profile is

$$\delta^* = \int_0^\delta \left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)\, dy = \frac{\delta}{3}$$

or

$$\frac{\delta^*}{x} = \frac{1.83}{\mathrm{Re}_x^{1/2}}$$

Integral Analysis for Laminar Boundary Layers (11)

Summary of results from von Kàrmàn integral analysis

Boundary layer thickness	$\dfrac{\delta}{x} = \dfrac{5.48}{\mathrm{Re}_x^{-1/2}}$
Momentum thickness	$\dfrac{\theta}{x} = \dfrac{0.73}{\mathrm{Re}_x^{-1/2}}$
Friction coefficient	$c_f = \dfrac{0.73}{\mathrm{Re}_x^{-1/2}}$

Blasius Analytical Solution for Laminar Boundary Layers (1)