MATLAB Programs for Computation of Isentropic Compressible Flow

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This document briefly summarizes a set of MATLAB programs for computing flow properties for the one-dimensional isentropic flow of an ideal gas. The nomenclature and sign conventions used here are consistent with the textbooks by Munson, Young and Okiishi [1], and White [2].

Governing Equations

Stagnation Properties as Functions of \( Ma \)

For one-dimensional, compressible, isentropic flow of and ideal gas the following equations relate the static properties, \( p, T, \) and \( \rho \) to the stagnation properties, \( p_0, T_0, \) and \( \rho_0. \)

\[
\frac{p}{p_0} = \left[ 1 + \frac{k - 1}{2} Ma^2 \right]^{k/(k-1)} \tag{1}
\]

\[
\frac{T}{T_0} = \frac{1}{1 + \frac{k - 1}{2} Ma^2} \tag{2}
\]

\[
\frac{\rho}{\rho_0} = \left[ 1 + \frac{k - 1}{2} Ma^2 \right]^{1/(k-1)} \tag{3}
\]

where \( k = c_p/c_v \) is the specific heat ratio and \( Ma = V/\sqrt{kRT} \) is the Mach number.

Duct Area Relationship for a Converging-Diverging Nozzle

For isentropic flow in a converging-diverging nozzle the ratio of local duct area to the area at the throat is uniquely related to the value of \( Ma. \) If \( A^* \) is the area of the duct section where \( Ma = 1, \) then the area at any other section along a converging-diverging nozzle is related to \( Ma \) by

\[
\frac{A}{A^*} = \frac{1}{Ma} \left[ 1 + \frac{k - 1}{2} Ma^2 \right]^{(k+1)/(2(k-1))} \tag{4}
\]
Note that this ratio may be computed even if the flow is not sonic at the minimum physical area. In that case $A^*$ is a reference value of the area, not the actual area of the duct at a particular section.

**Ma as a Function of Stagnation Properties**

If $Ma$ is unknown, but one of the preceding stagnation property ratios is known, then $Ma$ may be computed. Solving equation (1) through (3) for $Ma$ gives

\[
Ma = \sqrt{\frac{2}{k-1} \left[ \left( \frac{p}{p_0} \right)^{\frac{1-k}{k}} - 1 \right]}
\]  

(5)

\[
Ma = \sqrt{\frac{2}{k-1} \left[ \frac{T_0}{T} - 1 \right]}
\]  

(6)

\[
Ma = \sqrt{\frac{2}{k-1} \left[ \left( \frac{\rho}{\rho_0} \right)^{1-k} - 1 \right]}
\]  

(7)

**Ma as a Function of Area Ratio**

Equation (4) cannot be solved for $Ma$. If $A/A^*$ is known equation (4) can be used in a root-finding procedure to obtain a numerical value of $Ma$ that satisfies the equation. Rewriting equation (4) as

\[
f \left( \frac{A}{A^*} \right) = \frac{A}{A^*} - \frac{1}{Ma} \left[ 1 + \frac{k - 1}{2} \frac{Ma^2}{1 + \frac{k - 1}{2}} \right]^{(k+1)/(2(k-1))}
\]  

(8)

gives an equation suitable for use with the built-in `fzero` function. When the correct value of $A/A^*$ is guessed (for given values of $Ma$ and $k$) then $f(A/A^*) = 0$.

**M-files**

Table 1 lists the MATLAB functions that implement the computations outlined in the preceding equations.

**Examples**

Compute the property ratios $T/T_0$, $p/p_0$, and $\rho/\rho_0$ for air at $Ma = 0.75$.

\[
\text{>> isenTT0(0.75)}
\text{ans =}
\text{0.8989}
\]

\[
\text{>> isenpp0(0.75)}
\text{ans =}
\text{0.6886}
\]

\[
\text{>> isenrr0(0.75)}
\text{ans =}
\text{0.7660}
\]
Repeat the preceding calculations at \( Ma = 0.75 \) for Helium \((k = 1.66)\) instead of air

\[
\begin{align*}
\text{>> isenTT0}(0.75,1.66) & \\
\text{ans} & = 0.8434 \\
\text{>> isenpp0}(0.7,1.665) & \\
\text{ans} & = 0.6516 \\
\text{>> isenrr0}(0.75,1.66) & \\
\text{ans} & = 0.7726
\end{align*}
\]

Now, assume that the stagnation properties are known, but \( Ma \) is not. If \( T/T_0 = 0.5 \) for air, the value of \( Ma \) is

\[
\begin{align*}
\text{>> isenMaTT0}(0.5) & \\
\text{ans} & = 2.2361 \\
\text{>> isenTT0(ans)} & \quad \% \text{ check preceding calculation} \\
\text{ans} & = 0.5000
\end{align*}
\]

The call to \texttt{isenTT0} reverses the computation of \( Ma \), thereby providing a check on the calculations in the m-file. (See \texttt{testIsenProps} for a complete set of tests.)

For \( Ma = 0.75 \) and \( Ma = 1.5 \) the area ratio in equation (4) is computed with

\[
\begin{align*}
\text{>> isenAAs}(0.75) & \\
\text{ans} & = 1.0624 \\
\text{>> isenAAs}(1.5) & \\
\text{ans} & = 1.1762
\end{align*}
\]

The inverse computation is handled by the \texttt{isenMaas} function.

\[
\begin{align*}
\text{>> isenMaas}(1.0624) & \\
\text{ans} & = 0.7500 \\
\text{>> isenMaas}(1.1762) & \\
\text{ans} & = 0.6104
\end{align*}
\]

This last result appears to be in error, but it is not. By default, \texttt{isenMaas} returns the subsonic \( Ma \) that satisfies equation (4) for a given value of \( A/A^* \). For a given \( A/A^* \) both subsonic \((Ma < 1)\) and supersonic \((Ma > 1)\) solutions are possible. To select the supersonic solution a second input to the \texttt{isenMaas} function is needed. Only the sign of the second argument is important: if the second argument is negative the subsonic branch is chosen, if it is positive the supersonic branch is chosen. Thus

\[
\begin{align*}
\text{>> isenMaas}(1.1762,1) & \\
\text{ans} & = 1.5000
\end{align*}
\]
confirms that \texttt{isenAAs} and \texttt{isenMaas} are working correctly.

The \texttt{testIsenProps}, \texttt{MYO_11_38}, and \texttt{White_E9_3} functions provide additional examples of using the m-file functions in this toolbox.

## References


<table>
<thead>
<tr>
<th>Function</th>
<th>equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>aasmaResidual</td>
<td>(8)</td>
<td>Evaluates equation (8) for use with a root-finding procedure for finding $Ma$ as a function of $A/A^*$.</td>
</tr>
<tr>
<td>\texttt{MYO_11_38}</td>
<td>N.A.</td>
<td>Computations used in solution to problem 11.38 in Munson, Young and Okiishi.</td>
</tr>
<tr>
<td>\texttt{isenAAs}</td>
<td>(4)</td>
<td>Area ratio $A/A^*$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{isenMaas}</td>
<td>N.A.</td>
<td>$Ma$ as a function of area ratio $A/A^<em>$ for isentropic compressible flow. Computing $Ma$ requires a root-finding procedure so the equation for $A/A^</em>$ as a function of $Ma$ cannot be written explicitly. \texttt{isenMaas} uses the built-in \texttt{fzero} function and the \texttt{aasmaResidual} function to find the value of $A/A^<em>$ that gives $f(A/A^</em>) = 0$ in equation (8).</td>
</tr>
<tr>
<td>\texttt{isenMapp0}</td>
<td>(5)</td>
<td>$Ma$ as a function of pressure ratio $p/p_0$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{isenMarr0}</td>
<td>(7)</td>
<td>$Ma$ as a function of density ratio $\rho/\rho_0$ for isentropic compressible flow</td>
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<tr>
<td>\texttt{isenMaTT0}</td>
<td>(6)</td>
<td>$Ma$ as a function of temperature ratio $T/T_0$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{isenpp0}</td>
<td>(1)</td>
<td>Pressure ratio $p/p_0$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{isenrr0}</td>
<td>(3)</td>
<td>Density ratio $\rho/\rho_0$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{isenTT0}</td>
<td>(2)</td>
<td>Temperature ratio $T/T_0$ for isentropic compressible flow</td>
</tr>
<tr>
<td>\texttt{testIsenProps}</td>
<td>N.A.</td>
<td>Test all routines in this toolbox</td>
</tr>
<tr>
<td>\texttt{White_E9_3}</td>
<td>N.A.</td>
<td>Computations used in Example 9.3 in White.</td>
</tr>
</tbody>
</table>

Table 1: Functions for computing isentropic flow properties for one-dimensional compressible flow of an ideal gas.