EAS 199   Equations for a Breathing LED
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1 Goal: Equations for the Breathing LED Indicator

The final goal of this exercise is to develop an Arduino program to imitate the Apple LED indicator. In these notes, we discuss the procedure for obtaining the equations that describe the brightness of the LED as a function of time.

In practice, the brightness of an LED cannot be specified directly with an Arduino command[1]. Instead, the LED brightness is controlled indirectly by changing duty cycle of the PWM output that supplies power to the LED circuit. Focusing on the PWM output is a helpful simplification and is sufficient to create the breathing effect.

The PWM output is increased and decreased in a regular pattern during each breath cycle. The time variation of the PWM output is specified with a piecewise continuous mathematical function defined by two compact formulas. There are alternative strategies, e.g., table look-up, but using a piecewise continuous function provides a compact representation that can be easily modified.

1.1 Learning Objectives

These notes develop some simple formulas for the shape of the PWM curve used to drive the breathing LED. The learning objectives for this exercise are

1. Be able to define the characteristics of a piece-wise continuous function.

2. Be able to sketch the shape of the PWM output curves.

3. Be able to obtain the coefficients of an exponential function, given two data pairs that the function must pass through.

4. Be able to specify the two exponential functions that define the inhale and exhale portions of the breathing LED cycle.

These exercises presume that you already understand

- The meaning of exponential and logarithmic functions.

- How to use algebra to solve for two equations in two unknowns.

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[1]In fact "brightness" is not a rigorous term. In lighting technology the terms "luminance" (or luminosity) and "radiance" have rigorous definitions. Wikipedia has a useful introduction to these terms.

1.2 Pattern of the Breathing LED

The diagram to the right represents the pattern of PWM output to be used in the final implementation of the Arduino program. Specifically, this plot shows the duty cycle output from the `analogWrite` command.

The breathing pattern has three phases: inhale, pause, and exhale. The inhale and exhale phases are modeled with exponentially increasing and exponentially decreasing functions of time. The pattern repeats every $t_3 - t_0$ units of time. The inhale phase ends at t_1, and the exhale phase begins at t_2.
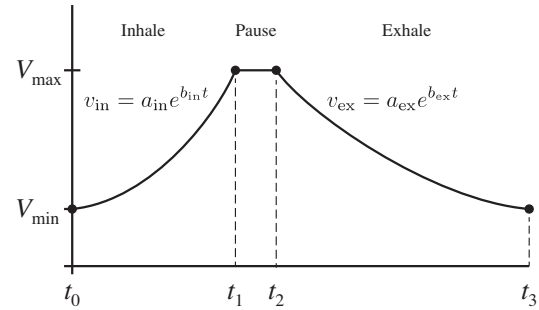
1.3 Piecewise functions

The breathing pattern repeats indefinitely, as suggested in the following diagram. We only need to define the function for the first complete cycle of inhale-pause-breath. The PWM output function is *piecewise-continuous*. The function is continuous because at no time is there a sudden change in the *value* of the function. Clearly the slope changes abruptly at each point marked by a solid dot. The function is called *piecewise* because the mathematical definition of the function (which has not yet been specified) changes along the t axis. In other words, not only is v a function of time, the definition of $v(t)$ is different for different time intervals. The boundary for the function definitions are marked by vertical dashed lines and solid dots on the $v(t)$ curve.
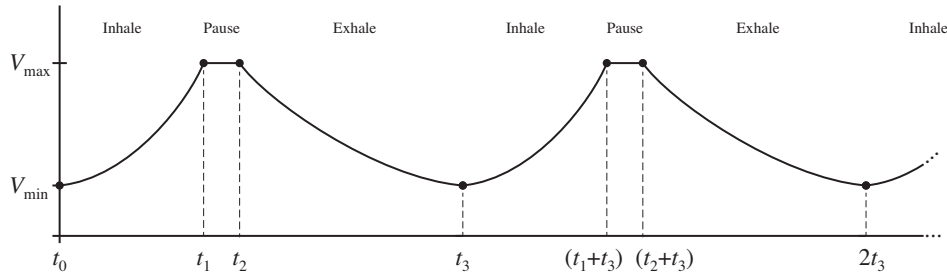
The piecewise function does not create any special difficulty for an Arduino program. The Arduino program needs a way to keep track of time so that the appropriate function for each interval can be selected.

The equations for the inhale and exhale phases are, respectively

$$v = a_{\mathrm{in}} e^{b_{\mathrm{in}} t} \qquad\qquad v = a_{\mathrm{ex}} e^{b_{\mathrm{ex}} t} \tag{1}$$

where v is the value sent to the PWM output. Remember that the values used in the `analogWrite` function must be limited to the range $0 \le v \le 255$.

The goal of these notes is to obtain formulas for a_{in}, b_{in}, a_{ex} and b_{ex} given specified values of v and t along the inhale and exhale curves. The equation for the pause phase is simple: $v = V_{\max}$, where $V_{\max}$ is a constant.

2 Using Two Points to Define the Coefficients of $v(t)$

The inhale and exhale functions are of the form

$$v = ae^{bt} \tag{2}$$

We require that this function pass through two points (t_A, v_A) and (t_B, v_B). Substituting these data pairs into Equation (2) gives

$$v_A = ae^{bt_A} \tag{3}$$
$$v_B = ae^{bt_B} \tag{4}$$

Points A and B can be any two pair of (t, v) values along the inhale and exhale curves. In practice, these points will be chosen as the endpoints inhale curve and the endpoints of the exhale curve.

Whereas Equation (2) is the $v(t)$ curve for any point along either the inhale or the exhale curve, Equations (3) and (4) apply to two specific data pairs. We assume that (t_A, v_A) and (t_B, v_B) are known. This leads us to interpret Equations (3) and (4) as conditions that determine the values for a and b. Therefore, by solving for the a and b that satisfy Equations (3) and (4), we obtain the coefficients of the function that passes through (t_A, v_A) and (t_B, v_B).

2.1 Linear Version of the Equation for Unknowns a and b

Finding the values of a and b that satisfy Equations (3) and (4) is the basic problem of solving two equations for two unknowns. The equations are non-linear, but in this case that slight problem is simplified by using the properties of the natural logarithm and the exponential function. Recall that if

$$z = xy$$

then

$$\ln(z) = \ln(x) + \ln(y).$$

In words: the logarithm of a product is the sum of the logarithms of the terms being multiplied. Also recall that if

$$r = e^{st}$$

then

$$\ln(r) = st.$$

In other words $\ln(\bullet)$ and $e^{(\bullet)}$ are each others inverse. If we take the logarithm of Equation (2) and apply the rules for manipulating logarithms of products, we get

$$\ln(v) = \ln\left[ae^{bt}\right] \quad \longrightarrow \quad \ln(v) = \ln(a) + \ln\left[e^{bt}\right] \quad \longrightarrow \quad \ln(v) = \ln(a) + bt$$

Applying that transformation to Equations (3) and (4) gives

$$\ln(v_A) = \ln(a) + bt_A \tag{5}$$
$$\ln(v_B) = \ln(a) + bt_B \tag{6}$$

These are two *linear* equations for the two unknowns, $\ln(a)$ and b. Now that we have transformed the system of nonlinear equations to a system of linear equations, it is a matter of simple algebra to solve for the a and b. Note that once we have a value of $\ln(a)$ we compute a with the exponential function, i.e., $\exp\left[\ln(a)\right] = a$.

2.2   Solving for a and b

Subtract Equation (6) from Equation (5) to get

$$\ln(v_A) - \ln(v_B) = b(t_A - t_B) \tag{7}$$

Since t_A, v_A, t_B and v_B are known, we can solve for b

$$b = \frac{\ln(v_A) - \ln(v_B)}{t_A - t_B}. \tag{8}$$

Now that the formula for b is known, we can substitute Equation (8) into either Equation (5) or Equation (6) to solve for a. This involves a bit more algebraic work, but the steps are straightforward

$$\ln(v_A) = \ln(a) + \left[\frac{\ln(v_A) - \ln(v_B)}{t_A - t_B}\right] t_A$$

$$\ln(a) = \ln(v_A) - \left[\frac{\ln(v_A) - \ln(v_B)}{t_A - t_B}\right] t_A$$

$$= \ln(v_A) \left[\frac{t_A - t_B}{t_A - t_B}\right] - \left[\frac{\ln(v_A) - \ln(v_B)}{t_A - t_B}\right] t_A$$

$$= \frac{\ln(v_A)\left[t_A - t_B\right] - \left[\ln(v_A) - \ln(v_B)\right] t_A}{t_A - t_B}$$

$$= \frac{t_A \ln(v_B) - t_B \ln(v_A)}{t_A - t_B}$$

where, in the last step the $+t_A \ln(v_A)$ and $-t_A \ln(v_A)$ terms in the numerator cancel each other. Applying the exponential function to both sides of the preceding equation gives the formula for computing a

$$a = \exp\left[\frac{t_A \ln(v_B) - t_B \ln(v_A)}{t_A - t_B}\right] \tag{9}$$

3   Applying the Equations

1. Choose appropriate values of $V_{\min}$, $V_{\max}$, t_1, t_2 and t_3. These are somewhat arbitrary *design* choices that you make to achieve a desired look to your inhale and exhale functions.

2. Use Equations (9) and (8) to compute a_{in} and b_{in}.

3. Use Equations (9) and (8) (again) to compute a_{ex} and b_{ex}.

Once you have obtained values for a_{in}, b_{in}, a_{ex} and b_{ex}, it is a good idea to add this step

4. Plot the $v_{\mathrm{in}}(t)$ and $v_{\mathrm{ex}}(t)$ functions (say, with Excel, MathCAD, or MATLAB) to make sure you do not have an error in your algebra.