

Computer Graphics

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<http://www.cs.pdx.edu/~fliu/courses/cs447/>

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Last time

Modeling Techniques

Today

- Project 2 grading schedule is out
 - Check your pdx.edu email
 - Before you come to the grading appointment, send a report to **both** fliu@pdx.edu and abhijay@pdx.edu
- More modeling techniques
- Splines

Implicit Functions

- Some surfaces can be represented as the vanishing points of functions (defined over 3D space)
 - Places where a function $f(x,y,z)=0$
 - Some objects are easy to represent this way
 - Spheres, ellipses, and similar
 - More generally, quadratic surfaces:
$$ax^2 + bx + cy^2 + dy + ez^2 + fz + g = 0$$
 - Shapes depends on all the parameters a,b,c,d,e,f,g
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Blobs and Metaballs

- Define the location of some points, \mathbf{p}_i
 - For each point, define a function on the distance to a given point, $D(\mathbf{x}, \mathbf{p}_i)$
 - Sum these functions up, and use them as an implicit function
 - Often, use Gaussian functions of distance, or other forms
 - Various results are called blobs or metaballs
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Example with Blobs



Rendered with POVray. Not everything is a blob, but the characters are.

Blob Math

- Implicit equation: $f(x, y, z) = w_0 + \sum_{i=1}^{n_{blobs}} w_i g_i(x, y, z) = 0$
- The w_i are weights - just numbers
- The g_i are functions, one common choice is:

$$g_i(\mathbf{x}) = e^{\frac{-\|\mathbf{x}-\mathbf{c}_i\|^2}{\sigma_i}}$$

- \mathbf{c}_i and σ_i are parameters
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Rendering Implicit Surfaces

- Some methods can render them directly
 - Raytracing - find intersections with Newton's method
 - For polygonal renderer, must convert to polygons
 - *Marching Cubes* algorithm
 - Also used for finding *iso-surfaces*, or *level sets*, in volume data
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Implicit Surfaces Summary

□ Advantages:

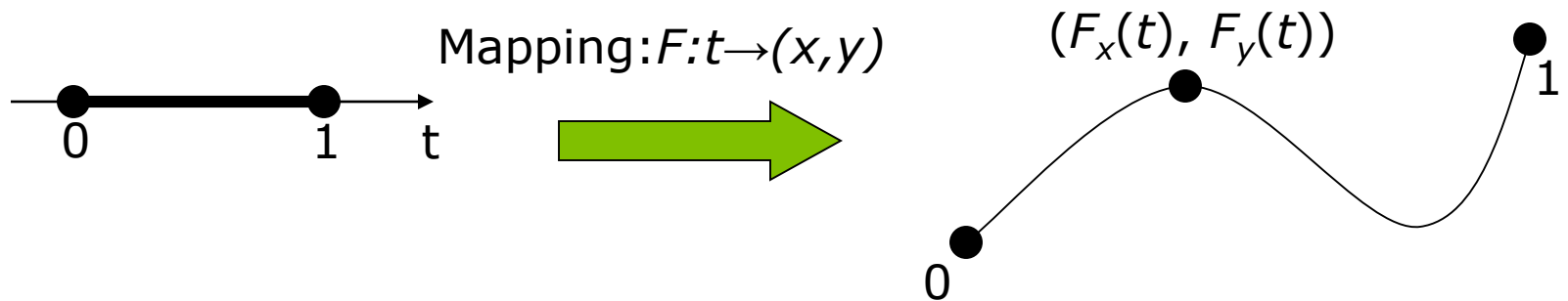
- Good for organic looking shapes, e.g. human body
- Good for extracting surfaces from volume representations, such as water surfaces in fluid simulation
- Easy inside/outside testing

□ Disadvantages:

- Difficult to render
 - Difficult to control when animating
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What are Parametric Curves?

- Define a parameter space
 - 1D for curves: t
 - 2D for surfaces: (s, t)
- Define a mapping from parameter space to 3D points
 - A function that takes parameter values and gives back 3D points
- The result is a parametric curve or surface



Why Parametric Curves?

- Parametric curves are intended to provide the generality of polygon meshes but with fewer parameters for smooth surfaces
 - Polygon meshes have as many parameters as there are vertices (at least)
 - Fewer parameters make it faster to create a curve, and easier to edit an existing curve
 - Normal vectors and texture coordinates can be easily defined everywhere
 - Parametric curves are easier to animate than polygon meshes
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Parametric Curves

- We have seen the parametric form for a line:

$$x = (1-t)x_0 + tx_1$$

$$y = (1-t)y_0 + ty_1$$

$$z = (1-t)z_0 + tz_1$$

- Note that x , y and z are each given by an equation that involves:
 - The parameter t
 - Some user specified control points, x_0 and x_1
 - This is an example of a parametric curve
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Basis Functions (first sighting)

- A line is the sum of two functions multiplied by vectors:

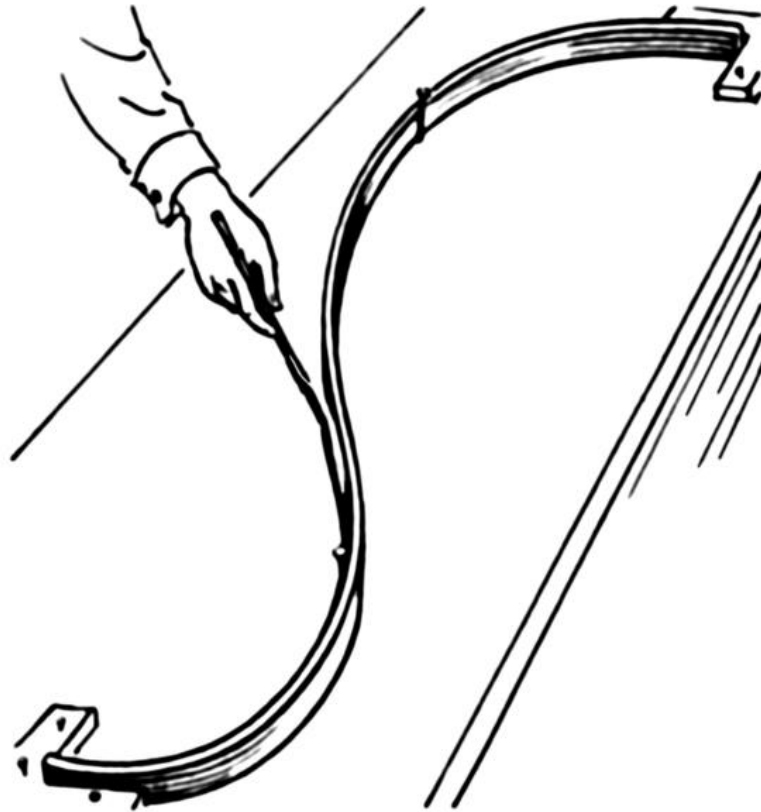
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1-t) \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

- A linear combination of *basis functions*
 - t and $1-t$ are the basis functions
 - The weights are called *control points*
 - (x_0, y_0, z_0) and (x_1, y_1, z_1) are the control points
 - They control the shape and position of the curve
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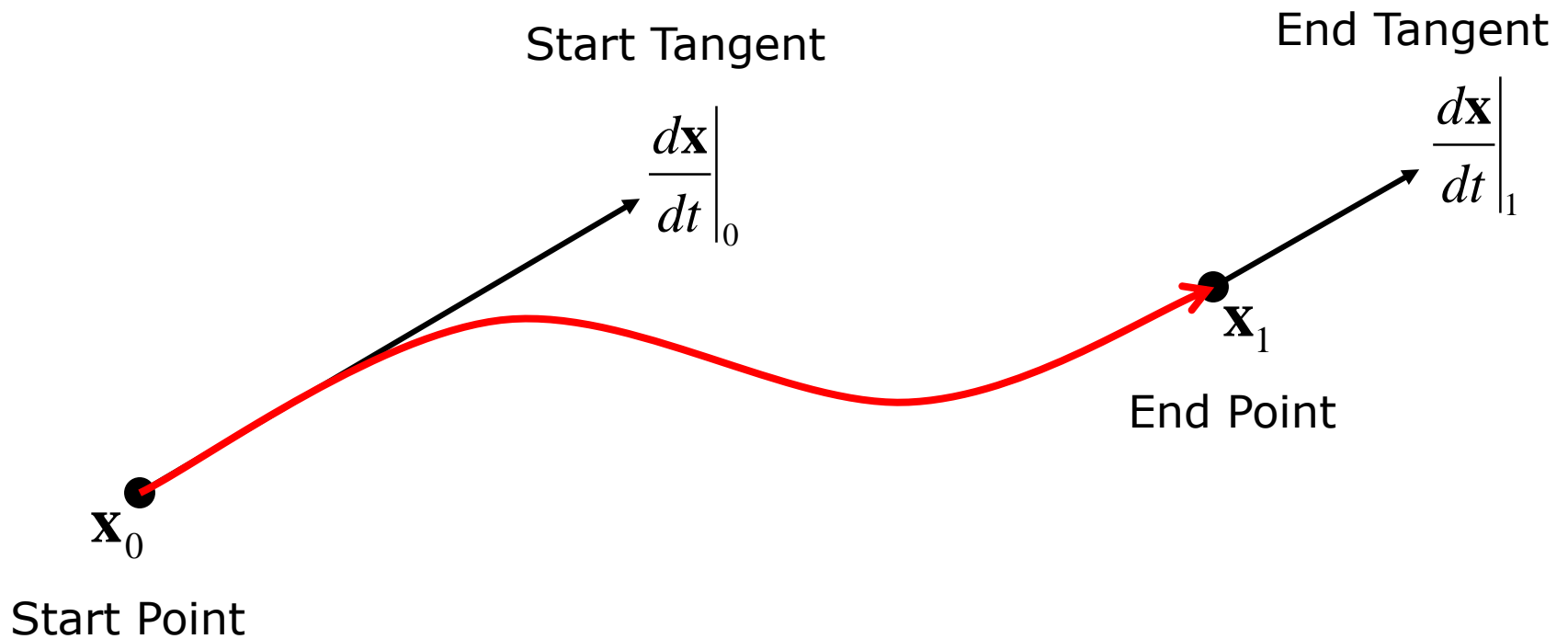
Hermite Spline

- A *spline* is a parametric curve defined by *control points*
 - The term spline dates from engineering drawing, where a spline was a piece of flexible wood used to draw smooth curves
 - The control points are *adjusted by the user* to control the shape of the curve
 - A *Hermite spline* is a curve for which the user provides:
 - The endpoints of the curve
 - The parametric derivatives of the curve at the endpoints (tangents with length)
 - The parametric derivatives are dx/dt , dy/dt , dz/dt
 - That is enough to define a *cubic* Hermite spline
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Spline



Control Point Interpretation



Hermite Spline (2)

□ Say the user provides $\mathbf{x}_0, \mathbf{x}_1, \left. \frac{d\mathbf{x}_0}{dt} \right|_0, \left. \frac{d\mathbf{x}_1}{dt} \right|_1$

□ A cubic spline has degree 3, and is of the form:

$$x = at^3 + bt^2 + ct + d$$

■ For some constants a, b, c and d derived from the control points, but how?

□ We have constraints:

■ The curve must pass through x_0 when $t=0$

■ The derivative must be x'_0 when $t=0$

■ The curve must pass through x_1 when $t=1$

■ The derivative must be x'_1 when $t=1$

Hermite Spline (3)

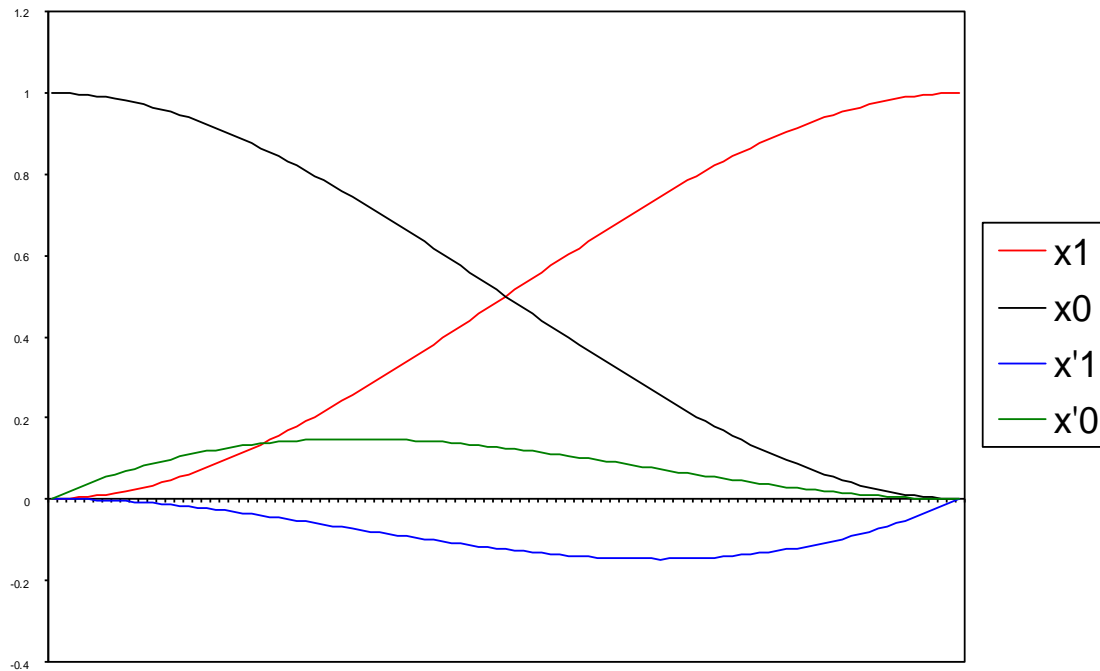
- Solving for the unknowns gives:
- $$\begin{aligned}a &= -2x_1 + 2x_0 + x'_1 + x'_0 \\b &= 3x_1 - 3x_0 - x'_1 - 2x'_0 \\c &= x'_0 \\d &= x_0\end{aligned}$$

- Rearranging gives:

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_1(-2t^3 + 3t^2) \\ &+ \mathbf{x}_0(2t^3 - 3t^2 + 1) \\ &+ \mathbf{x}'_1(t^3 - t^2) \\ &+ \mathbf{x}'_0(t^3 - 2t^2 + t)\end{aligned} \quad \text{or} \quad x = \begin{bmatrix} x_1 & x_0 & x'_1 & x'_0 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Basis Functions

- A point on a Hermite curve is obtained by multiplying each control point by some function and summing



Splines in 2D and 3D

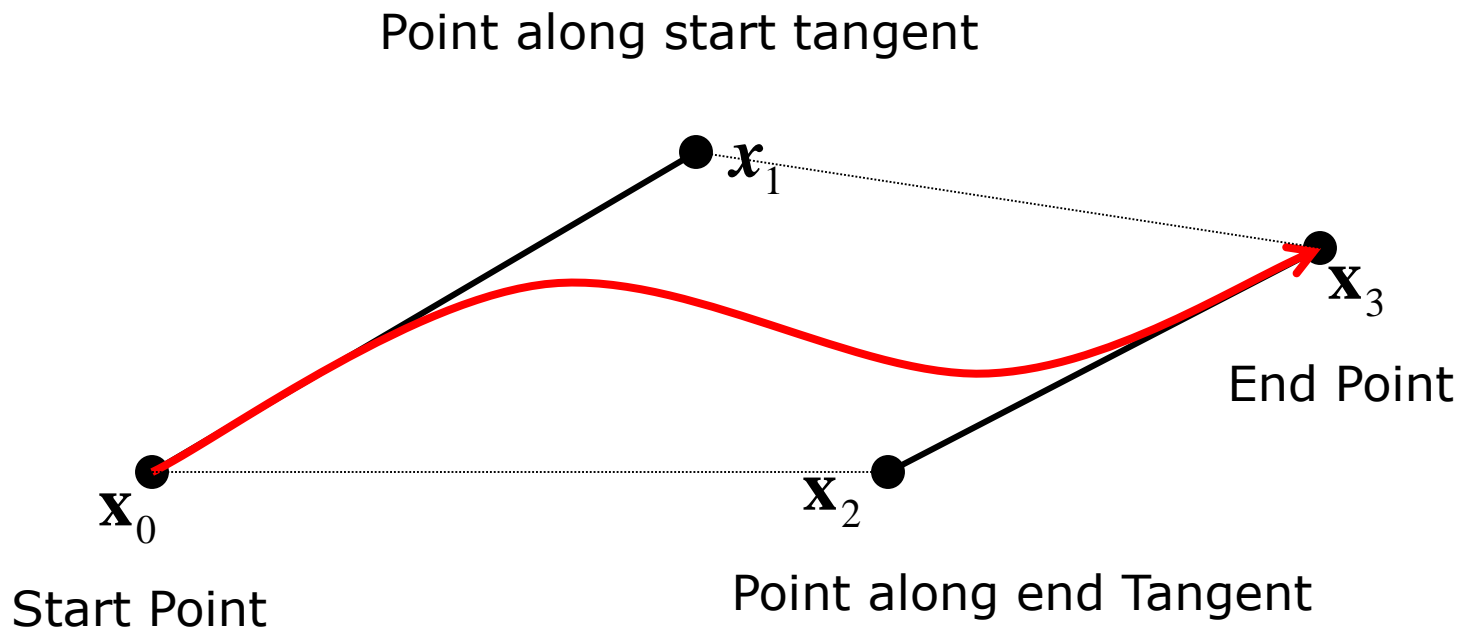
- For higher dimensions, define the control points in higher dimensions (that is, as vectors)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_0 & x'_1 & x'_0 \\ y_1 & y_0 & y'_1 & y'_0 \\ z_1 & z_0 & z'_1 & z'_0 \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Bezier Curves (1)

- Different choices of basis functions give different curves
 - Choice of basis determines how the control points influence the curve
 - In Hermite case, two control points define endpoints, and two more define parametric derivatives
 - For Bezier curves, two control points define endpoints, and two control the tangents at the endpoints in a geometric way
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Control Point Interpretation



Bezier Curves (2)

□ The user supplies $d+1$ control points, \mathbf{p}_i

□ Write the curve as:

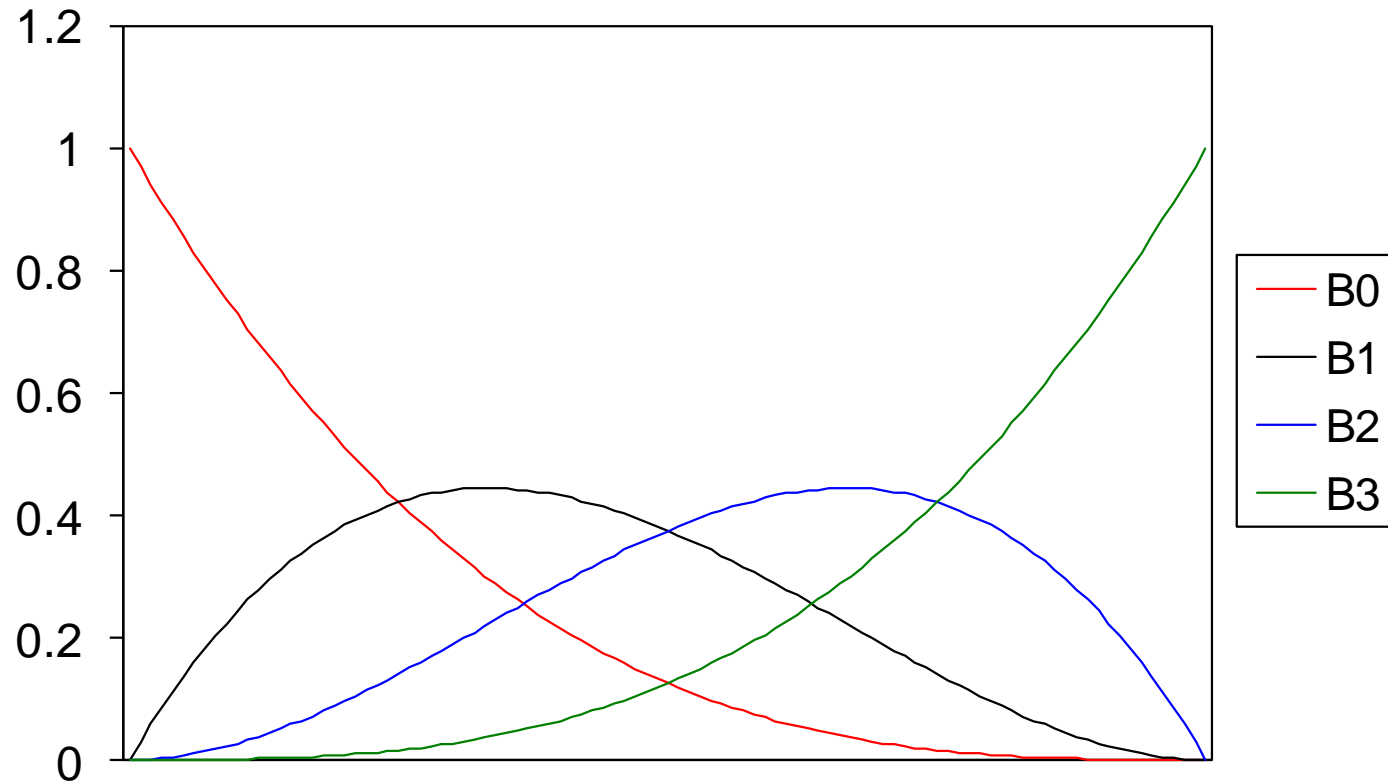
$$\mathbf{x}(t) = \sum_{i=0}^d \mathbf{p}_i B_i^d(t) \qquad B_i^d(t) = \binom{d}{i} t^i (1-t)^{d-i}$$

□ The functions B_i^d are the *Bernstein polynomials* of degree d

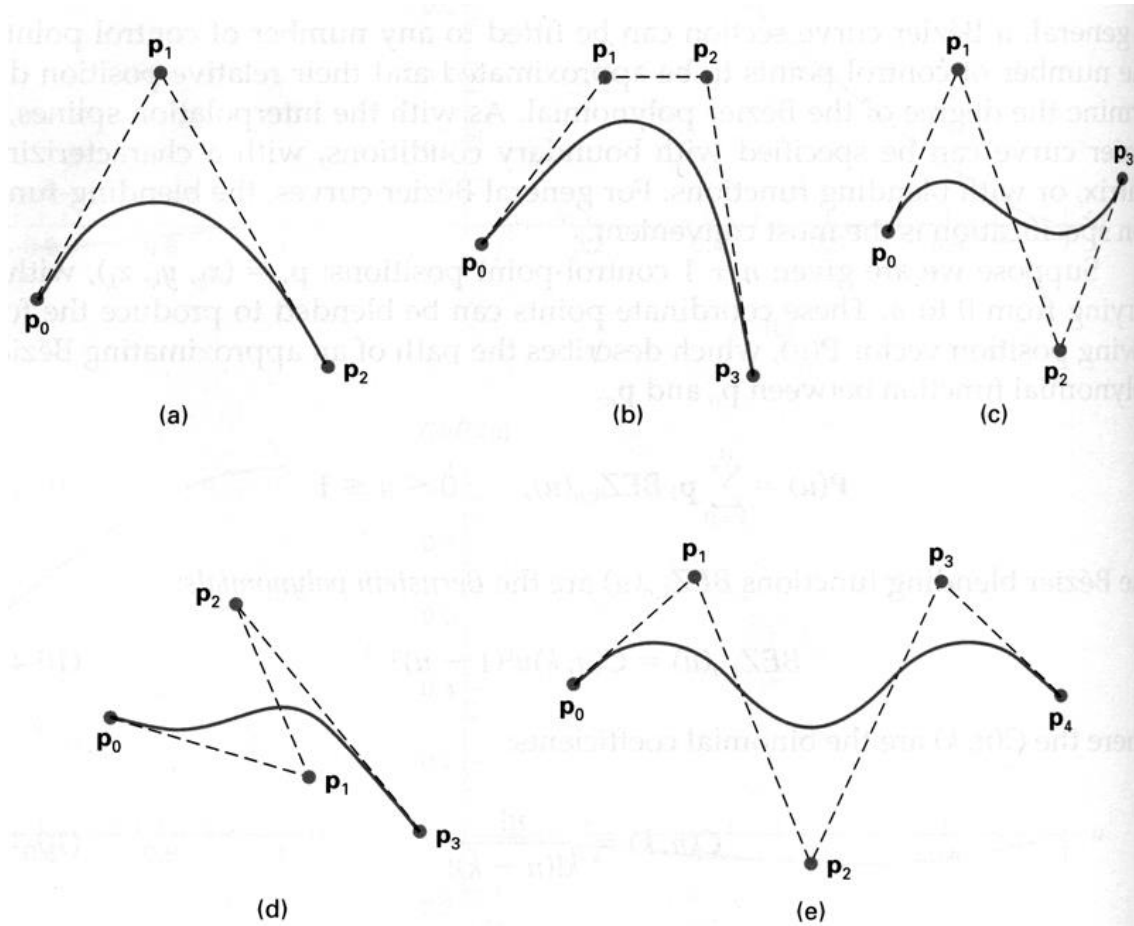
□ This equation can be written as a matrix equation also

- There is a matrix to take Hermite control points to Bezier control points
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Bezier Basis Functions for $d=3$



Bezier Curves of Varying Degree

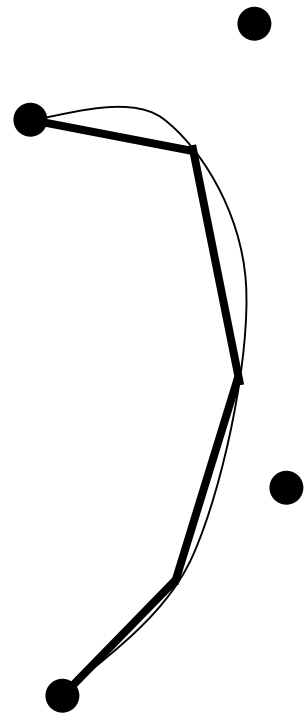


Bezier Curve Properties

- The first and last control points are interpolated
 - The tangent to the curve at the first control point is along the line joining the first and second control points
 - The tangent at the last control point is along the line joining the second last and last control points
 - The curve lies entirely within the convex hull of its control points
 - The Bernstein polynomials (the basis functions) sum to 1 and are everywhere positive
 - They can be rendered in many ways
 - E.g.: Convert to line segments with a subdivision algorithm
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Rendering Bezier Curves (1)

- Evaluate the curve at a fixed set of parameter values and join the points with straight lines
- Advantage: Very simple
- Disadvantages:
 - Expensive to evaluate the curve at many points
 - No easy way of knowing how fine to sample points, and maybe sampling rate must be different along curve
 - No easy way to adapt. In particular, it is hard to measure the deviation of a line segment from the exact curve



Rendering Bezier Curves (2)

- Recall that a Bezier curve lies entirely within the convex hull of its control vertices
 - If the control vertices are nearly collinear, then the convex hull is a good approximation to the curve
 - Also, a cubic Bezier curve can be *subdivided* into two shorter curves that exactly cover the original
 - This suggests an algorithm:
 - Keep breaking the curve into sub-curves
 - Stop when the control points of each sub-curve are nearly collinear
 - Draw the control polygon - the polygon formed by the control points
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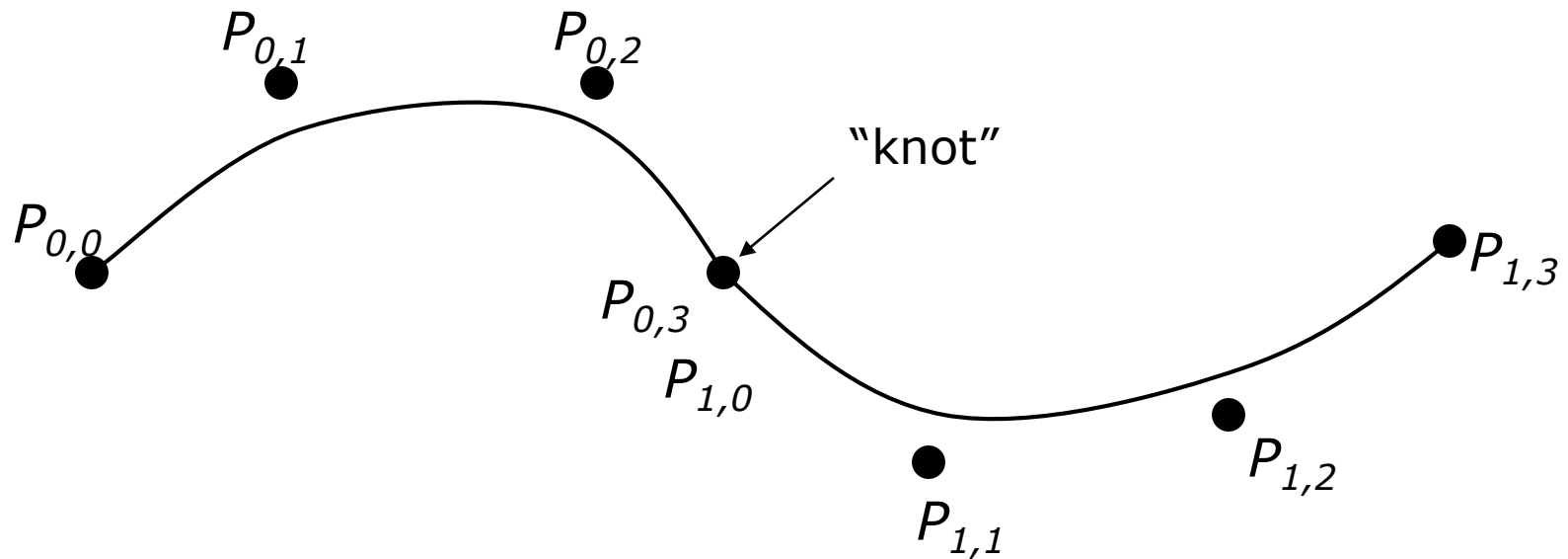
Invariance

- *Translational invariance* means that translating the control points and then evaluating the curve is the same as evaluating and then translating the curve
 - *Rotational invariance* means that rotating the control points and then evaluating the curve is the same as evaluating and then rotating the curve
 - These properties are essential for parametric curves used in graphics
 - It is easy to prove that Bezier curves, Hermite curves and everything else we will study are translation and rotation invariant
 - Some forms of curves, *rational splines*, are also *perspective invariant*
 - Can do perspective transform of control points and *then* evaluate the curve
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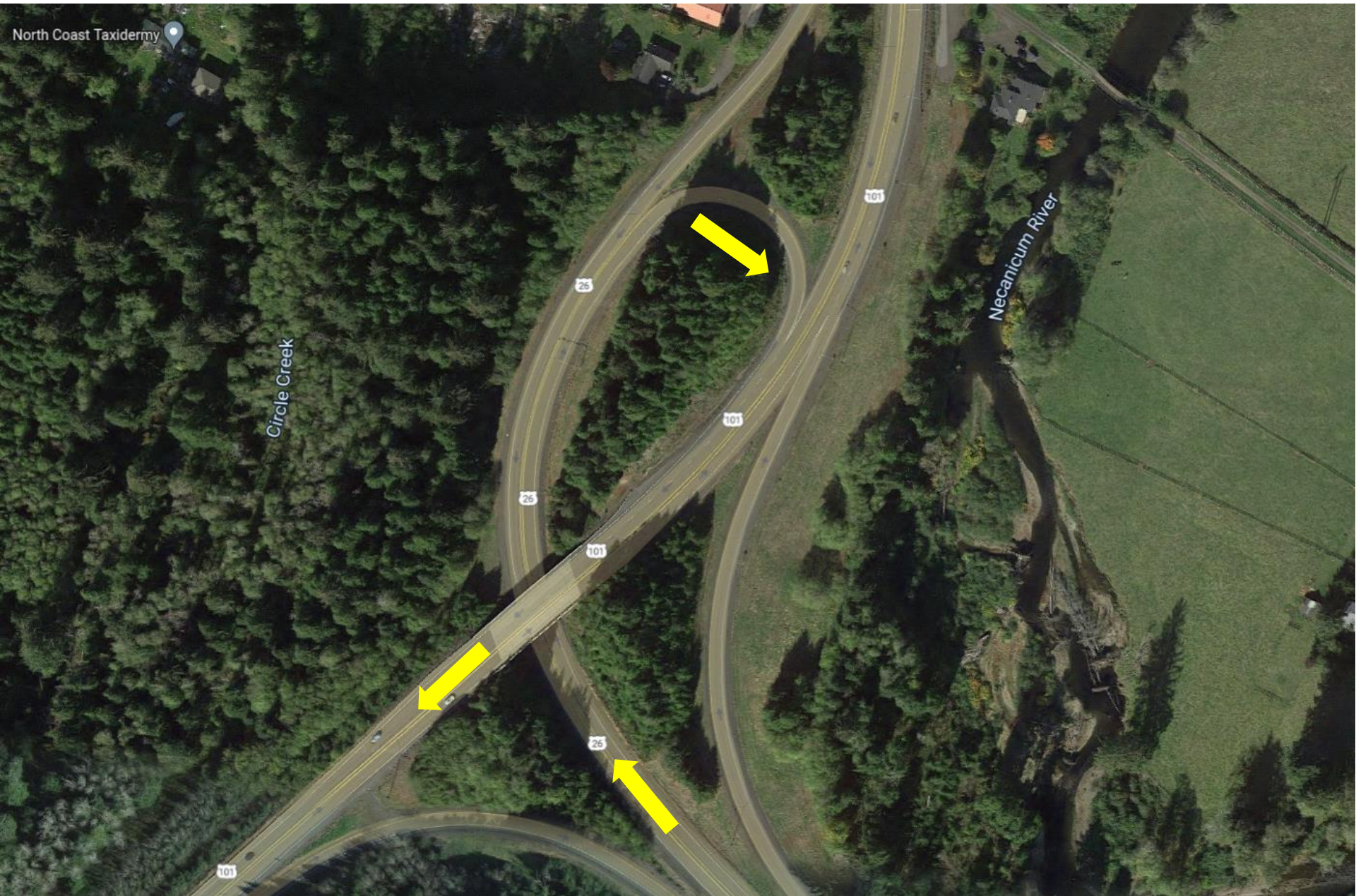
Longer Curves

- A single cubic Bezier or Hermite curve can only capture a small class of curves
 - One solution is to raise the degree
 - Allows more control, at the expense of more control points and higher degree polynomials
 - Control is not *local*, one control point influences entire curve
 - Alternate, most common solution is to join pieces of cubic curve together into *piecewise cubic curves*
 - Total curve can be broken into pieces, each of which is cubic
 - *Local control*. Each control point only influences a limited part of the curve
 - Interaction and design is much easier
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Piecewise Bezier Curve



North Coast Taxidermy



Circle Creek

Necanicum River

101

26

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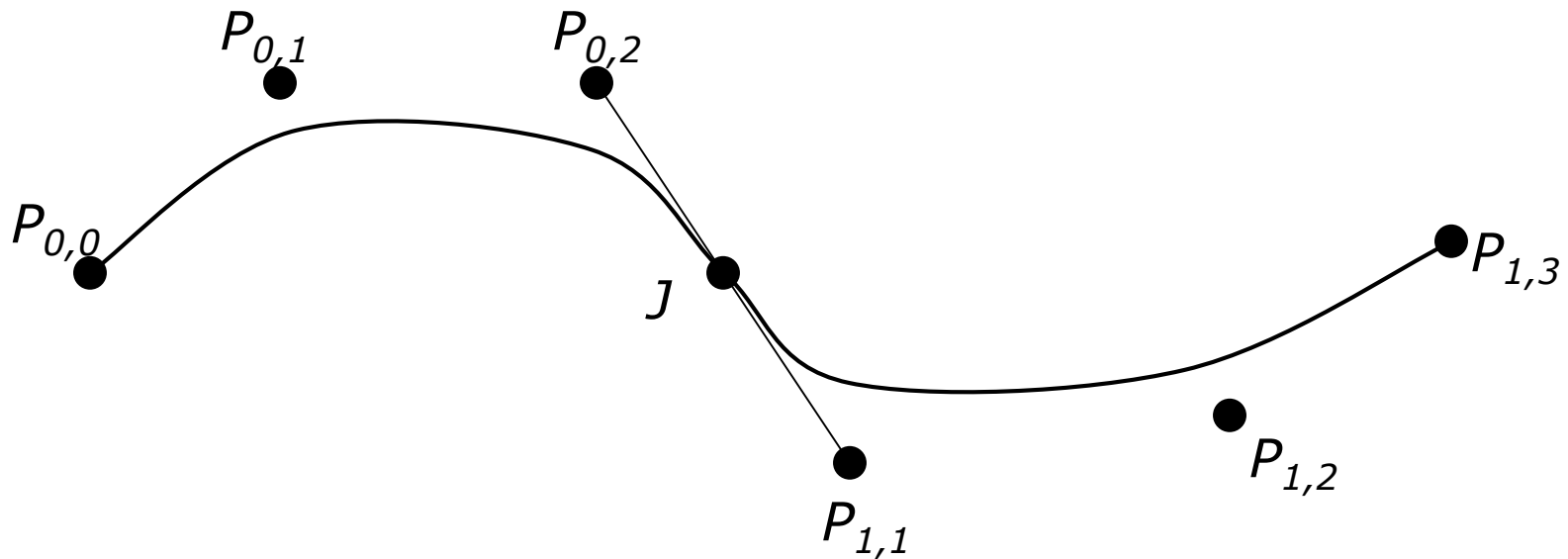
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Continuity

- When two curves are joined, we typically want some degree of continuity across the boundary (the knot)
 - C^0 , “C-zero”, point-wise continuous, curves share the same point where they join
 - C^1 , “C-one”, continuous derivatives, curves share the same parametric derivatives where they join
 - C^2 , “C-two”, continuous second derivatives, curves share the same parametric second derivatives where they join
 - Higher orders possible
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Bezier Continuity



Disclaimer: PowerPoint curves are not Bezier curves, they are interpolating piecewise quadratic curves! This diagram is an approximation.

Sketch of Proof for C¹

Bezier curve equation:

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_0(1-t)^3 + \mathbf{x}_1 3t(1-t)^2 + \mathbf{x}_2 3t^2(1-t) + \mathbf{x}_3 t^3 \\ &= \mathbf{x}_0(1-3t+3t^2-t^3) + \mathbf{x}_1 3(t-2t^2+t^3) + \mathbf{x}_2 3(t^2-t^3) + \mathbf{x}_3 t^3\end{aligned}$$

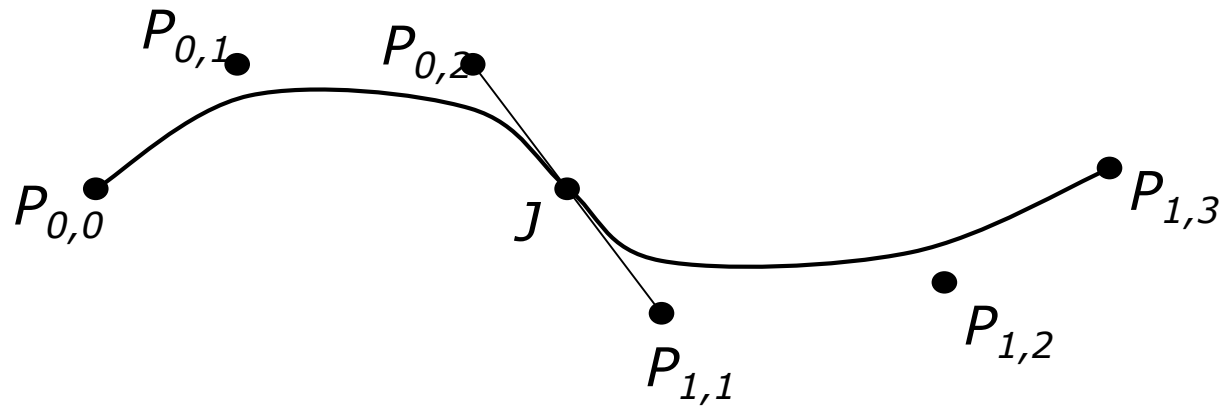
Parametric derivative:

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_0(-3+6t-3t^2) + \mathbf{x}_1 3(1-4t+3t^2) + \mathbf{x}_2 3(2t-3t^2) + \mathbf{x}_3 3t^2$$

Evaluated at endpoint of curve (note proves tangent property):

$$\left. \frac{d\mathbf{x}}{dt} \right|_{t=0} = -3\mathbf{x}_0 + 3\mathbf{x}_1 = 3(\mathbf{x}_1 - \mathbf{x}_0) \qquad \left. \frac{d\mathbf{x}}{dt} \right|_{t=1} = -3\mathbf{x}_2 + 3\mathbf{x}_3 = 3(\mathbf{x}_3 - \mathbf{x}_2)$$

Proof (cont)



C^1 requires equal parametric derivatives:

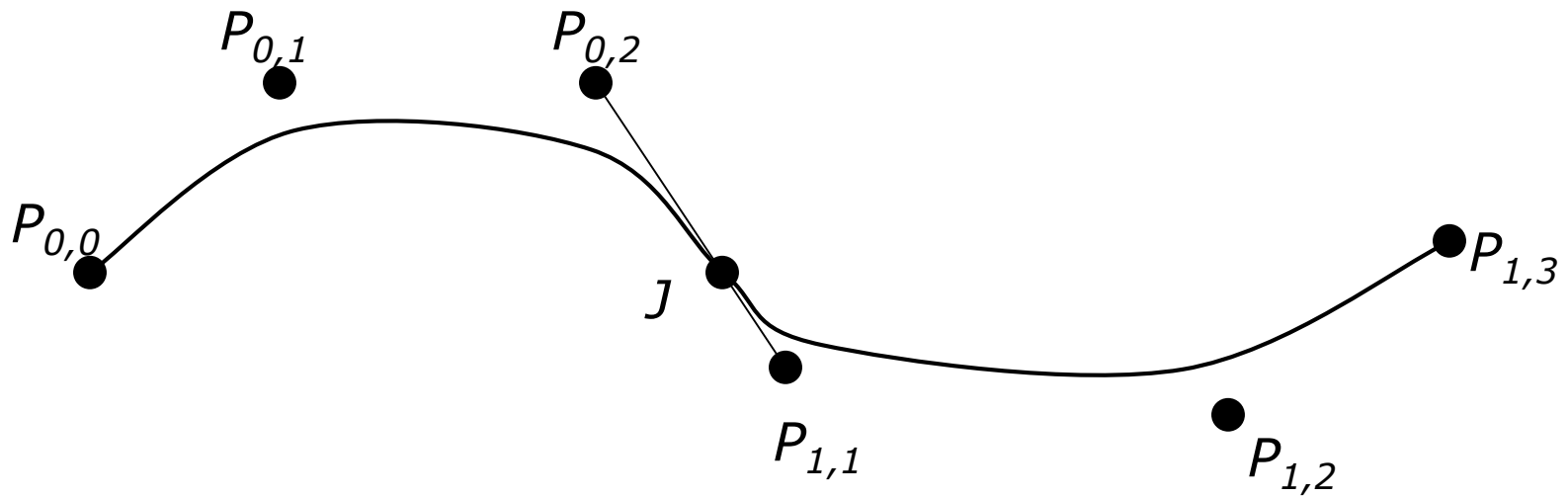
$$3(\mathbf{P}_{1,1} - \mathbf{J}) = 3(\mathbf{J} - \mathbf{P}_{0,2})$$

$$\mathbf{P}_{1,1} - \mathbf{J} = \mathbf{J} - \mathbf{P}_{0,2}$$

Geometric Continuity

- Derivative continuity is important for animation
 - If an object moves along the curve with constant parametric speed, there should be no sudden jump at the knots
 - For other applications, *tangent continuity* might be enough
 - Requires that the tangents point in the same direction
 - Referred to as G^1 *geometric continuity*
 - Curves *could* be made C^1 with a re-parameterization: $u=f(t)$
 - The geometric version of C^2 is G^2 , based on curves having the same radius of curvature across the knot
 - What is the tangent continuity constraint for a Bezier curve?
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Bezier Geometric Continuity



$$(\mathbf{P}_{1,1} - \mathbf{J}) = k(\mathbf{J} - \mathbf{P}_{0,2}) \quad \text{for some } k$$

Bezier Curve/Surface Problems

- To make a long continuous curve with Bezier segments requires using many segments
 - Same for large surface
- Maintaining continuity requires constraints on the control point positions
 - The user cannot arbitrarily move control vertices and automatically maintain continuity
 - The constraints must be explicitly maintained
 - It is not intuitive to have control points that are not free

B-splines

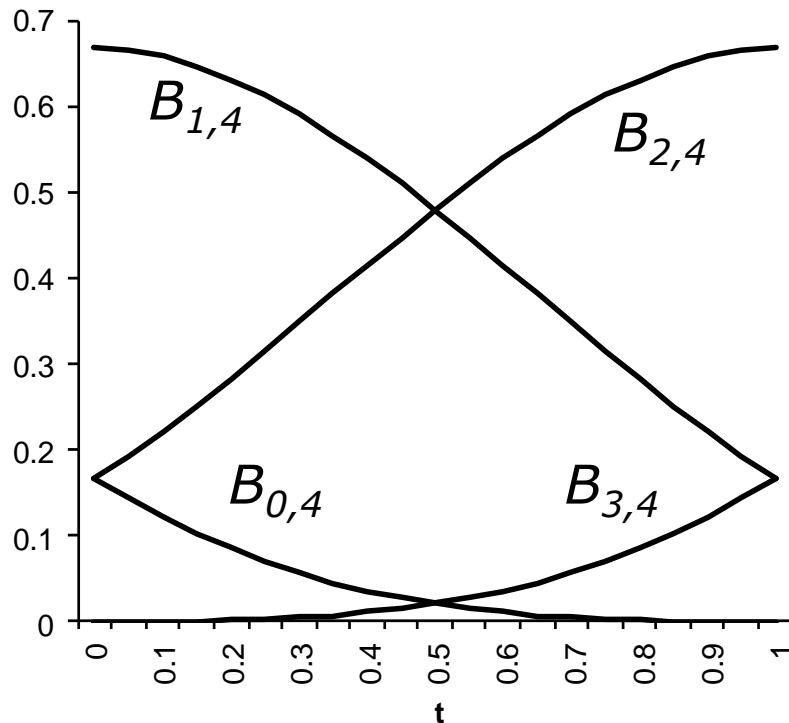
- B-splines automatically take care of continuity, with exactly one control vertex per curve segment
- Many types of B-splines: degree may be different (linear, quadratic, cubic,...) and they may be uniform or non-uniform
 - We will only look closely at uniform B-splines
- With uniform B-splines, continuity is always one degree lower than the degree of each curve piece
 - Linear B-splines have C^0 continuity, cubic have C^2 , etc

Uniform Cubic B-spline on $[0, 1)$

- Four control points are required to define the curve for $0 \leq t < 1$ (t is the parameter)
 - Not surprising for a cubic curve with 4 degrees of freedom
- The equation looks just like a Bezier curve, but with different basis functions
 - Also called *blending functions* - they describe how to blend the control points to make the curve (see Shirley book ch. 15.6)

$$\begin{aligned}x(t) &= \sum_{i=0}^3 P_i B_{i,4}(t) \\ &= P_0 \frac{1}{6} (1 - 3t + 3t^2 - t^3) + P_1 \frac{1}{6} (4 - 6t^2 + 3t^3) + P_2 \frac{1}{6} (1 + 3t + 3t^2 - 3t^3) + P_3 \frac{1}{6} (t^3)\end{aligned}$$

Basis Functions on $[0, 1]$



- Does the curve interpolate its endpoints?
- Does it lie inside its convex hull?

$$\begin{aligned}x(t) = & P_0 \frac{1}{6} (1 - 3t + 3t^2 - t^3) \\ & + P_1 \frac{1}{6} (4 - 6t^2 + 3t^3) \\ & + P_2 \frac{1}{6} (1 + 3t + 3t^2 - 3t^3) \\ & + P_3 \frac{1}{6} (t^3)\end{aligned}$$

Uniform Cubic B-spline on $[0, 1)$

- The blending functions sum to one, and are positive everywhere
 - The curve lies inside its convex hull
- The curve does not interpolate its endpoints
 - Requires hacks or non-uniform B-splines
- There is also a matrix form for the curve:

$$x(t) = \frac{1}{6} \begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Demo



BspIne-open.exe

Uniform B-spline at Arbitrary t

- The interval from an *integer* parameter value v to $v+1$ is essentially the same as the interval from 0 to 1
 - The parameter value is offset by v
 - A different set of control points is needed
- To evaluate a uniform cubic B-spline at an arbitrary parameter value t :
 - Find the greatest integer less than or equal to t : $v = \text{floor}(t)$
 - Evaluate:
$$X(t) = \sum_{i=0}^3 P_{i+v} B_{i,4}(t-v)$$
- Valid parameter range: $0 \leq t < n-3$, where n is the number of control points

Loops

- To create a loop, use control points from the start of the curve when computing values at the end of the curve:

$$X(t) = \sum_{i=0}^3 P_{(i+v) \bmod n} B_{i,4}(t-v)$$

- Any parameter value is now valid
 - Although for numerical reasons it is sensible to keep it within a small multiple of n

Demo



Bspine-closed.exe

B-splines and Interpolation, Continuity

- Uniform B-splines do not interpolate control points, unless:
 - You repeat a control point three times
 - But then all derivatives also vanish ($=0$) at that point
- To align tangents, use double control vertices
 - Then tangent aligns similar to Bezier curve
- Uniform B-splines are automatically C^2

How to Choose a Spline

- Hermite curves are good for single segments where you know the parametric derivative or want easy control of it
- Bezier curves are good for single segments or patches where a user controls the points
- B-splines are good for large continuous curves and surfaces

Next Time

Raytracing