Computer Graphics

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http://www.cs.pdx.edu/~fliu/courses/cs447/

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Last time

- Modeling Techniques

Today

- More modeling techniques
- Splines
- Homework 5 available, due 11/29 in class
- Pick slots for grading Project 2
  - https://docs.google.com/spreadsheets/d/1V0_oNMcXHx3D5zgY42ogDclmPaVuhp7oosYgMYyp-yg/edit?ts=5bf3968d#gid=0

Implicit Functions

- Some surfaces can be represented as the vanishing points of functions (defined over 3D space)
  - Places where a function $f(x,y,z)=0$
- Some objects are easy to represent this way
  - Spheres, ellipses, and similar
  - More generally, quadratic surfaces:

$$ax^2 + bx + cy^2 + dy + ez^2 + fz + g = 0$$

  - Shapes depends on all the parameters a,b,c,d,e,f,g

Blobs and Metaballs

- Define the location of some points, $\mathbf{p}_i$
- For each point, define a function on the distance to a given point, $D(\mathbf{x},\mathbf{p}_i)$
- Sum these functions up, and use them as an implicit function
- Often, use Gaussian functions of distance, or other forms
  - Various results are called blobs or metaballs

Example with Blobs

Rendered with POVray. Not everything is a blob, but the characters are.

Blob Math

- Implicit equation: $f(x,y,z) = w_0 + \sum_{i=1}^{n_{blobs}} w_i g_i(x,y,z) = 0$
- The w_i are weights - just numbers
- The g_i are functions, one common choice is:

$$g_i(\mathbf{x}) = e^{\frac{-\|\mathbf{x}-\mathbf{c}_i\|^2}{\sigma_i}}$$

  - $\mathbf{c}_i$ and σ_i are parameters

Rendering Implicit Surfaces

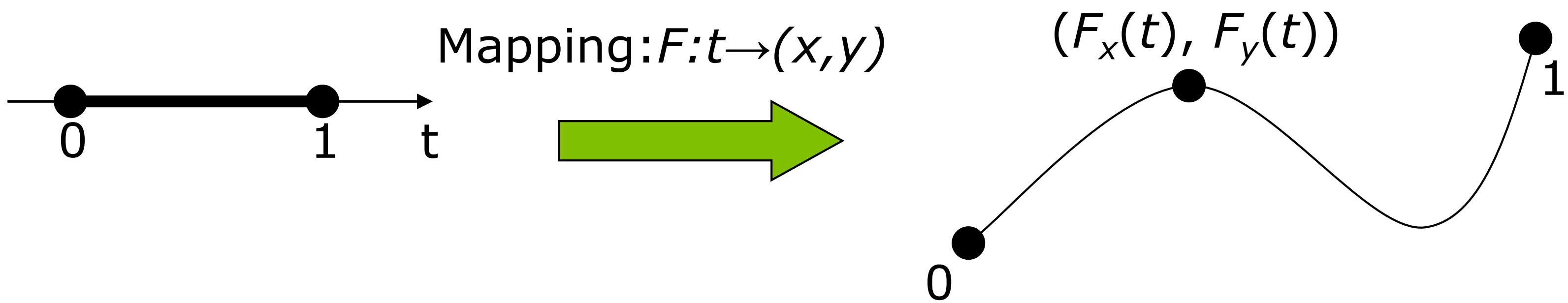
- Some methods can render them directly
  - Raytracing - find intersections with Newton’s method
- For polygonal renderer, must convert to polygons
  - *Marching Cubes* algorithm
  - Also used for finding *iso-surfaces*, or *level sets*, in volume data

Implicit Surfaces Summary

- Advantages:
  - Good for organic looking shapes eg human body
  - Good for extracting surfaces from volume representations, such as water surfaces in fluid simulation
  - Easy inside/outside testing
- Disadvantages:
  - Difficult to render
  - Difficult to control when animating

What are Parametric Curves?

- Define a parameter space
  - 1D for curves: *t*
  - 2D for surfaces: (*s*,*t*)
- Define a mapping from parameter space to 3D points
  - A function that takes parameter values and gives back 3D points
- The result is a parametric curve or surface



Why Parametric Curves?

- Parametric curves are intended to provide the generality of polygon meshes but with fewer parameters for smooth surfaces
  - Polygon meshes have as many parameters as there are vertices (at least)
- Fewer parameters make it faster to create a curve, and easier to edit an existing curve
- Normal vectors and texture coordinates can be easily defined everywhere
- Parametric curves are easier to animate than polygon meshes

Parametric Curves

- We have seen the parametric form for a line:

$$x = (1-t)x_0 + tx_1$$
$$y = (1-t)y_0 + ty_1$$
$$z = (1-t)z_0 + tz_1$$

- Note that x, y and z are each given by an equation that involves:
  - The parameter *t*
  - Some user specified control points, x_0 and x_1
- This is an example of a parametric curve

Basis Functions (first sighting)

- A line is the sum of two functions multiplied by vectors:

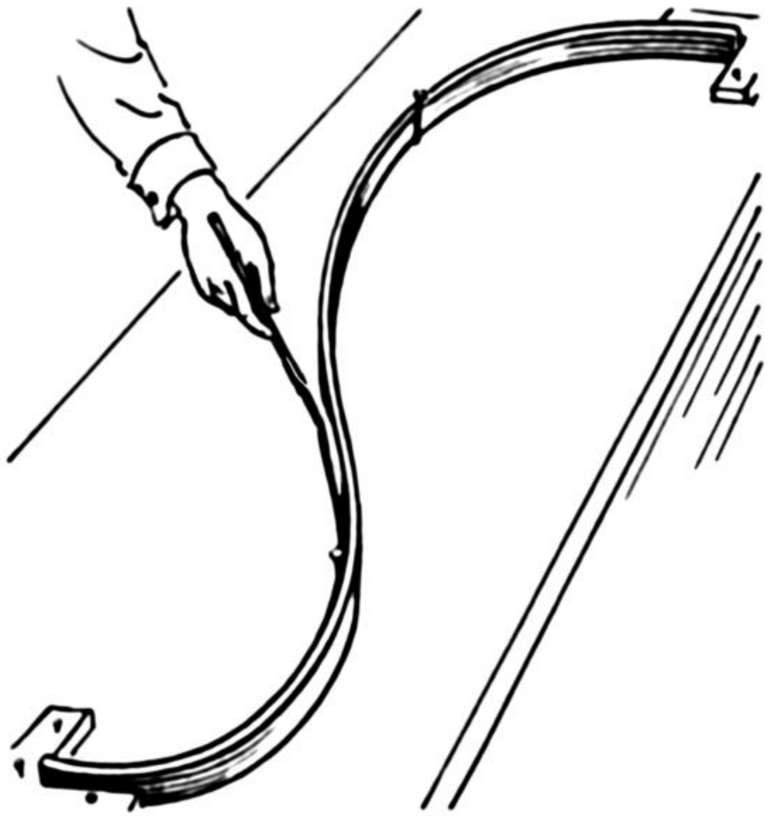
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1-t)\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

- A linear combination of *basis functions*
  - t and 1-t are the basis functions
- The weights are called *control points*
  - (x_0,y_0,z_0) and (x_1,y_1,z_1) are the control points
  - They control the shape and position of the curve

Hermite Spline

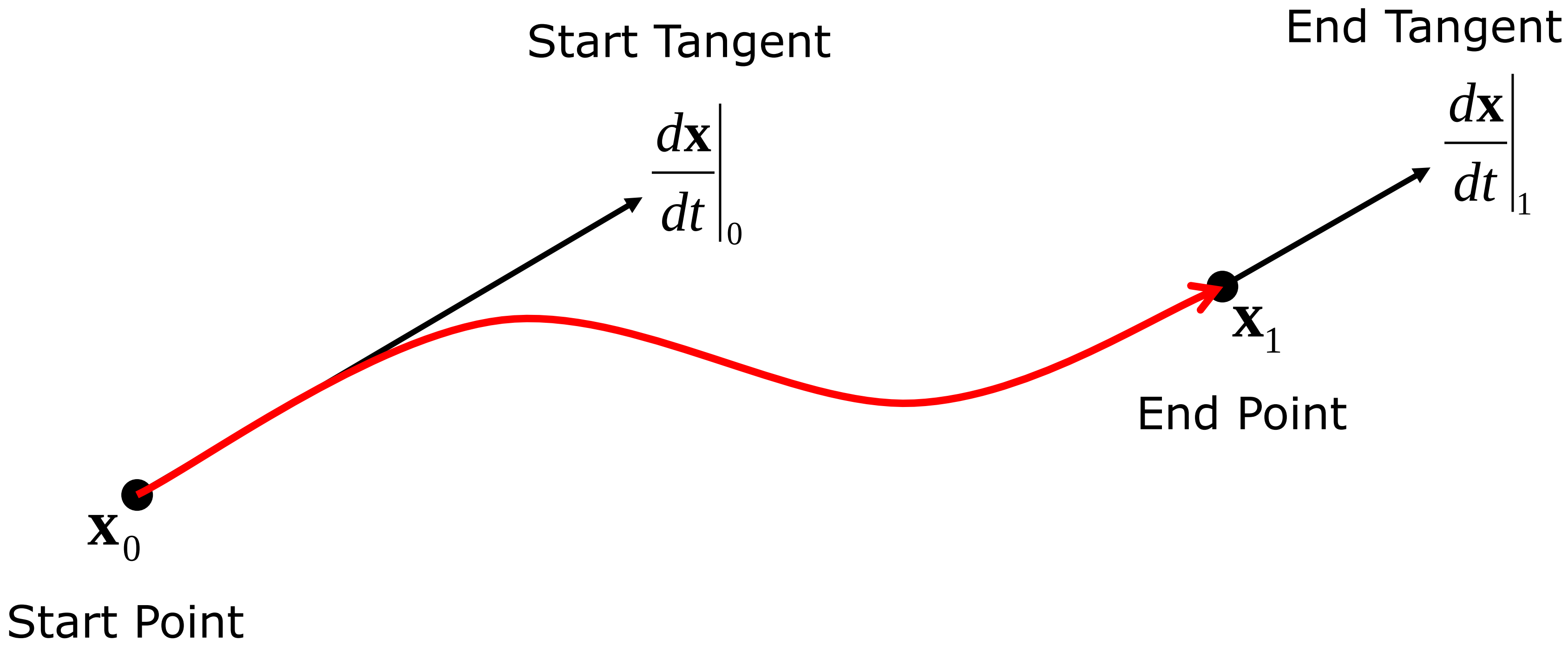
- A *spline* is a parametric curve defined by *control points*
  - The term spline dates from engineering drawing, where a spline was a piece of flexible wood used to draw smooth curves
  - The control points are *adjusted by the user* to control the shape of the curve
- A *Hermite spline* is a curve for which the user provides:
  - The endpoints of the curve
  - The parametric derivatives of the curve at the endpoints (tangents with length)
    - The parametric derivatives are *dx/dt, dy/dt, dz/dt*
  - That is enough to define a *cubic* Hermite spline

Spline



Source: http://en.wikipedia.org/wiki/File:Spline_(PSF).png

Control Point Interpretation



Hermite Spline (2)

- Say the user provides $\mathbf{x}_0, \mathbf{x}_1, \left.\frac{d\mathbf{x}_0}{dt}\right|_0, \left.\frac{d\mathbf{x}_1}{dt}\right|_1$
- A cubic spline has degree 3, and is of the form:

$$x = at^3 + bt^2 + ct + d$$

  - For some constants a, b, c and d derived from the control points, but how?
- We have constraints:
  - The curve must pass through x_0 when $t=0$
  - The derivative must be x'_0 when $t=0$
  - The curve must pass through x_1 when $t=1$
  - The derivative must be x'_1 when $t=1$

Hermite Spline (3)

- Solving for the unknowns gives:

$$a = -2x_1 + 2x_0 + x_1' + x_0'$$
$$b = 3x_1 - 3x_0 - x_1' - 2x_0'$$
$$c = x_0'$$
$$d = x_0$$

- Rearranging gives:

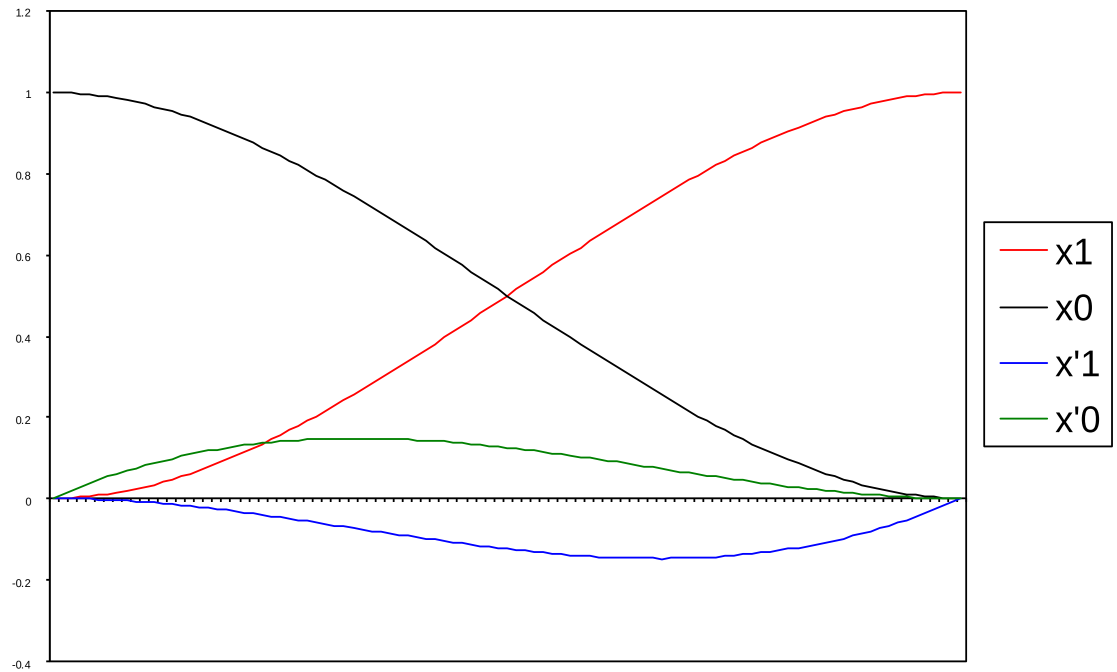
$$\begin{aligned}\mathbf{x} = &\mathbf{x}_1(-2t^3 + 3t^2) \\ &+ \mathbf{x}_0(2t^3 - 3t^2 + 1) \\ &+ \mathbf{x}_1'(t^3 - t^2) \\ &+ \mathbf{x}_0'(t^3 - 2t^2 + t)\end{aligned}$$

or

$$x = \begin{bmatrix} x_1 & x_0 & x_1' & x_0' \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Basis Functions

- A point on a Hermite curve is obtained by multiplying each control point by some function and summing



Splines in 2D and 3D

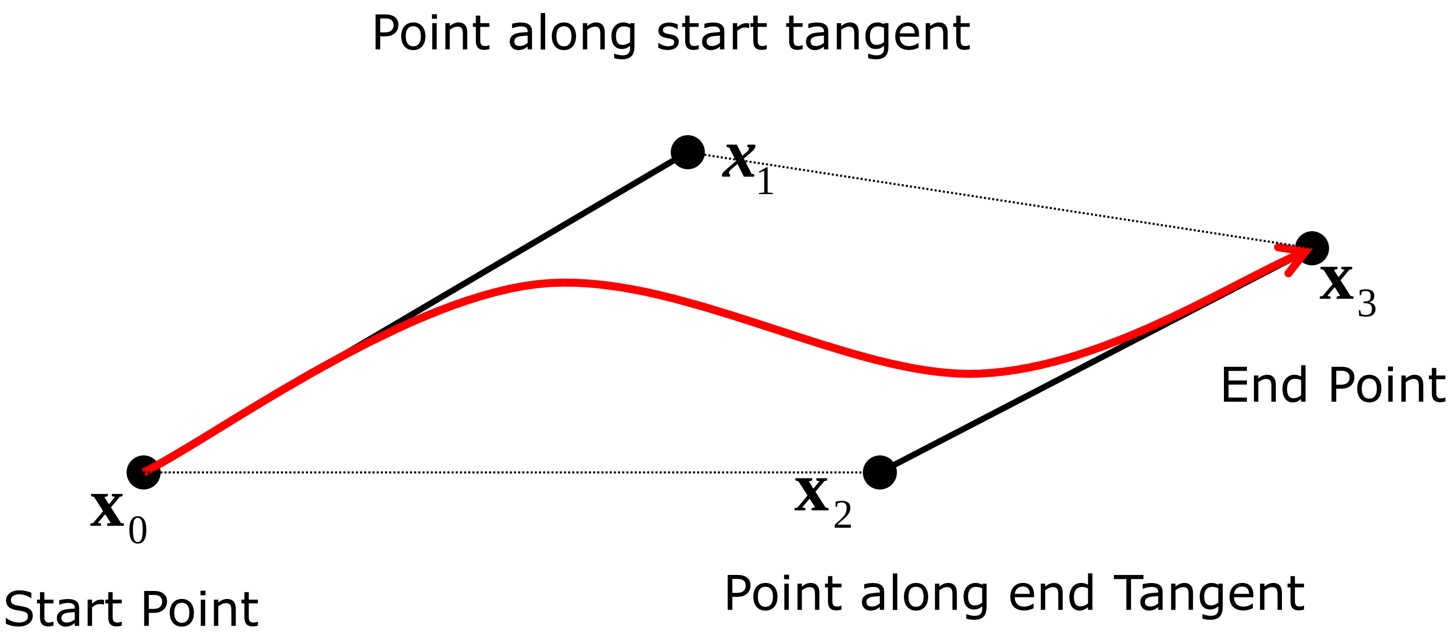
- For higher dimensions, define the control points in higher dimensions (that is, as vectors)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 & x_0 & x_1' & x_0' \\ y_1 & y_0 & y_1' & y_0' \\ z_1 & z_0 & z_1' & z_0' \end{bmatrix} \begin{bmatrix} -2 & 3 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Bezier Curves (1)

- Different choices of basis functions give different curves
  - Choice of basis determines how the control points influence the curve
  - In Hermite case, two control points define endpoints, and two more define parametric derivatives
- For Bezier curves, two control points define endpoints, and two control the tangents at the endpoints in a geometric way

Control Point Interpretation



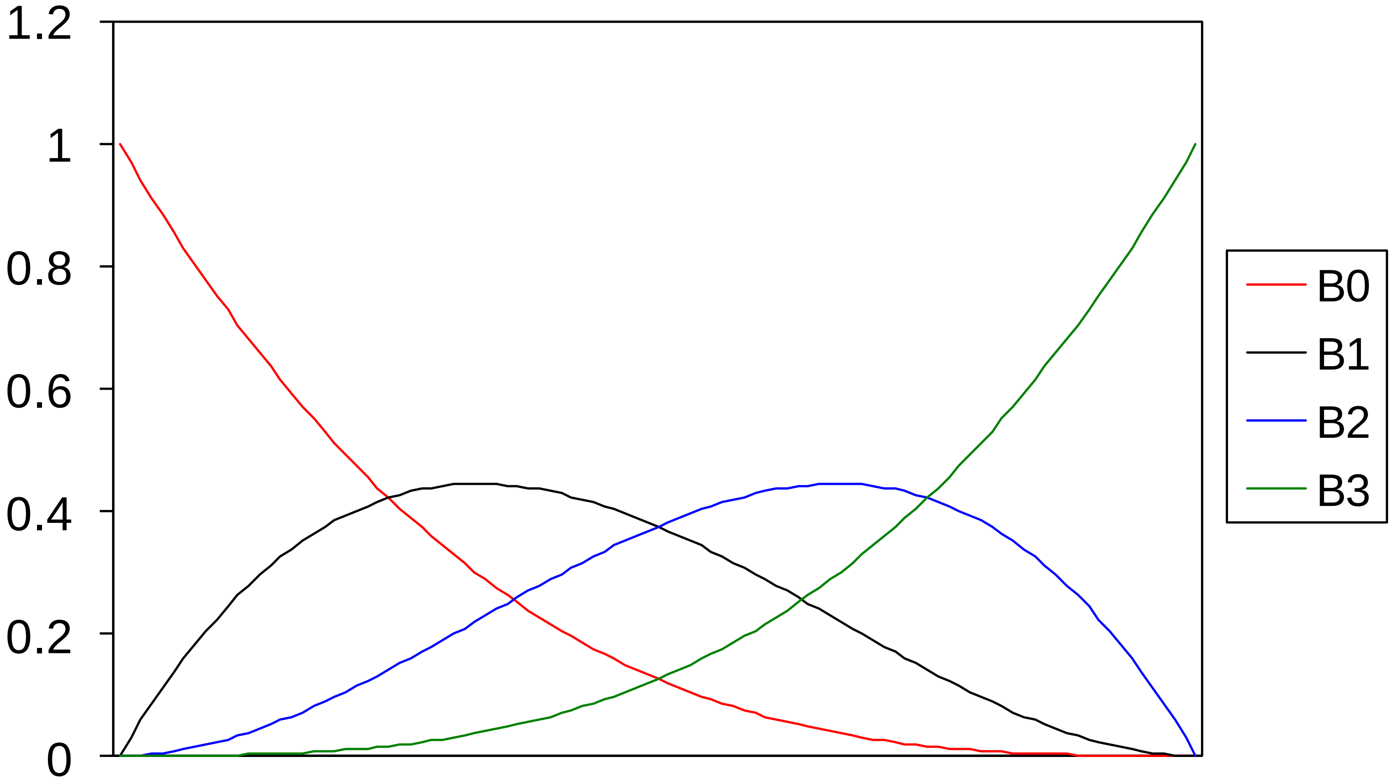
Bezier Curves (2)

- The user supplies $d+1$ control points, $\boldsymbol{p}_i$
- Write the curve as:

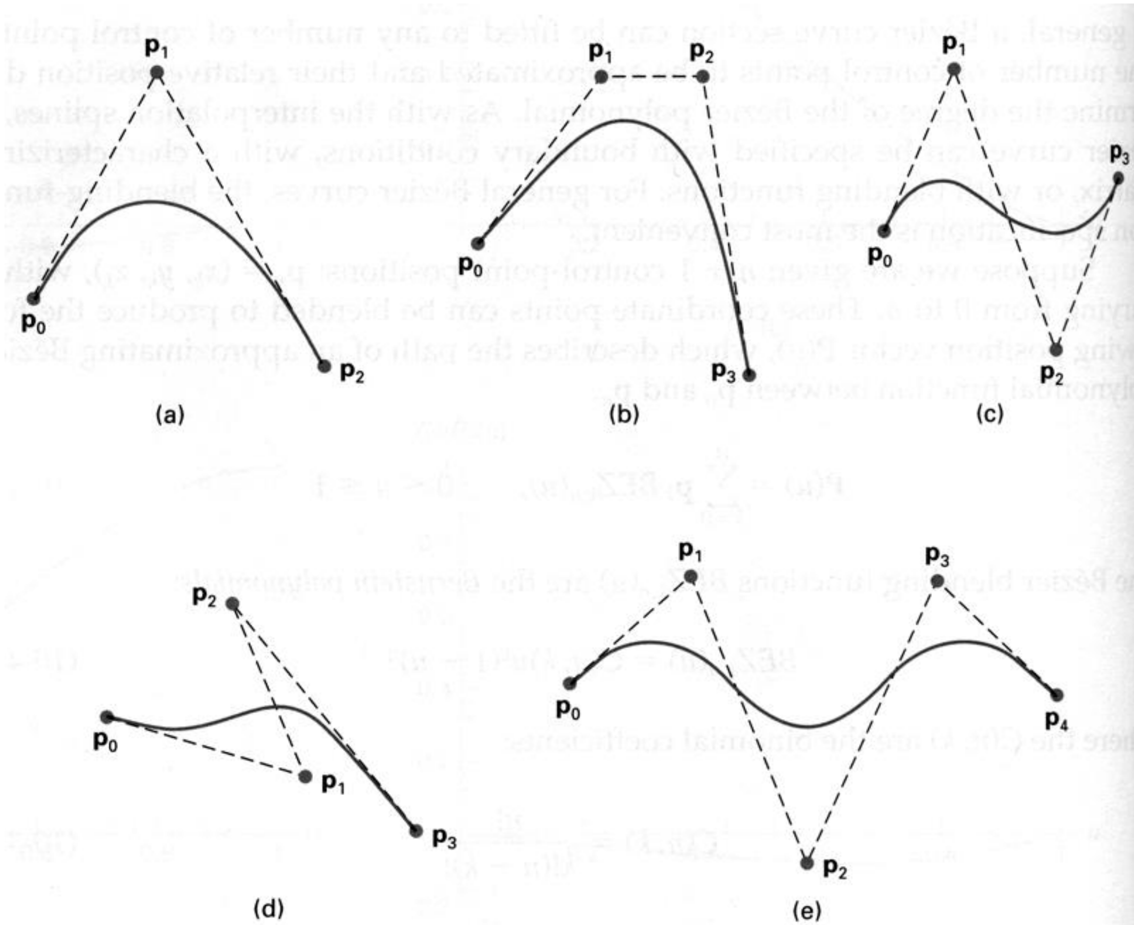
$$\mathbf{x}(t) = \sum_{i=0}^{d} \mathbf{p}_i B_i^d(t) \qquad B_i^d(t) = \binom{d}{i} t^i (1-t)^{d-i}$$

- The functions B_i^d are the *Bernstein polynomials* of degree d
- This equation can be written as a matrix equation also
  - There is a matrix to take Hermite control points to Bezier control points

Bezier Basis Functions for *d*=3



Bezier Curves of Varying Degree

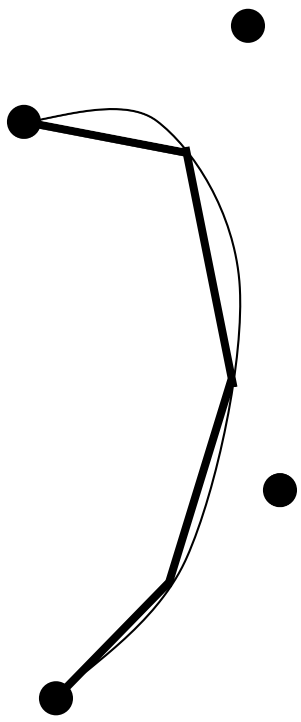


Bezier Curve Properties

- The first and last control points are interpolated
- The tangent to the curve at the first control point is along the line joining the first and second control points
- The tangent at the last control point is along the line joining the second last and last control points
- The curve lies entirely within the convex hull of its control points
  - The Bernstein polynomials (the basis functions) sum to 1 and are everywhere positive
- They can be rendered in many ways
  - E.g.: Convert to line segments with a subdivision algorithm

Rendering Bezier Curves (1)

- Evaluate the curve at a fixed set of parameter values and join the points with straight lines
- Advantage: Very simple
- Disadvantages:
  - Expensive to evaluate the curve at many points
  - No easy way of knowing how fine to sample points, and maybe sampling rate must be different along curve
  - No easy way to adapt. In particular, it is hard to measure the deviation of a line segment from the exact curve



Rendering Bezier Curves (2)

- Recall that a Bezier curve lies entirely within the convex hull of its control vertices
- If the control vertices are nearly collinear, then the convex hull is a good approximation to the curve
- Also, a cubic Bezier curve can be *subdivided* into two shorter curves that exactly cover the original
- This suggests an algorithm:
  - Keep breaking the curve into sub-curves
  - Stop when the control points of each sub-curve are nearly collinear
  - Draw the control polygon - the polygon formed by the control points

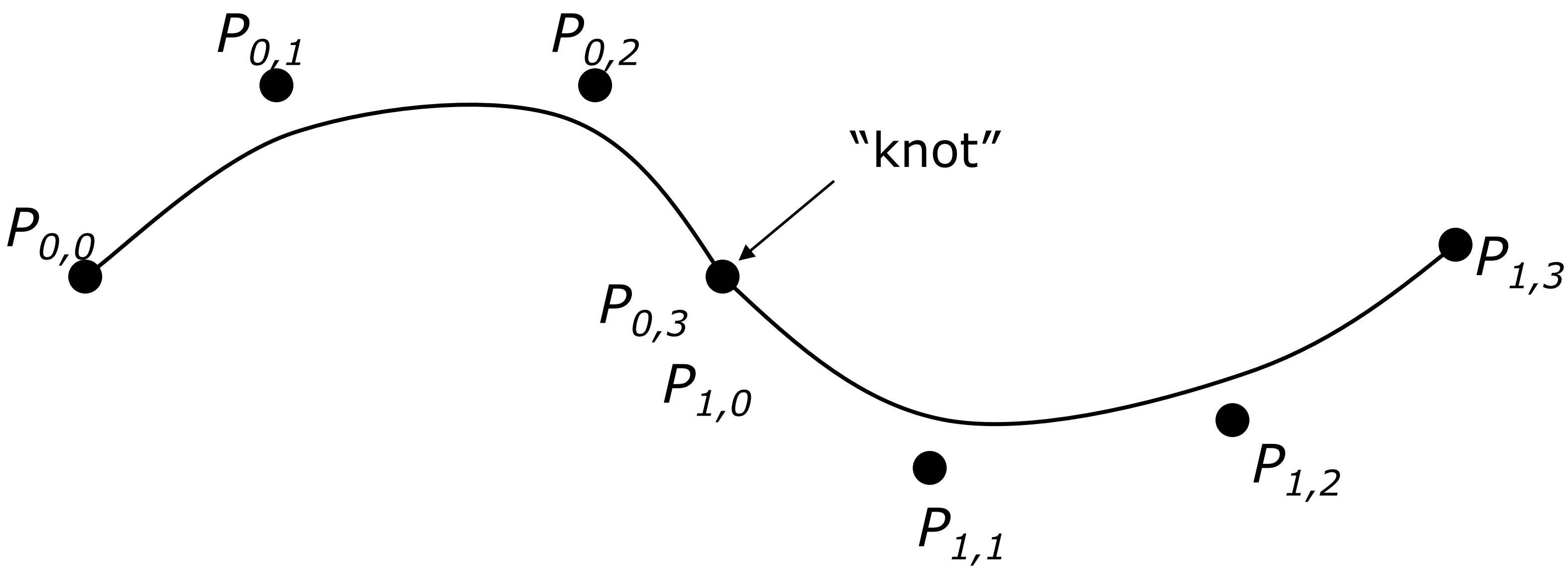
Invariance

- *Translational invariance* means that translating the control points and then evaluating the curve is the same as evaluating and then translating the curve
- *Rotational invariance* means that rotating the control points and then evaluating the curve is the same as evaluating and then rotating the curve
- These properties are essential for parametric curves used in graphics
- It is easy to prove that Bezier curves, Hermite curves and everything else we will study are translation and rotation invariant
- Some forms of curves, *rational splines*, are also *perspective invariant*
  - Can do perspective transform of control points and *then* evaluate the curve

Longer Curves

- A single cubic Bezier or Hermite curve can only capture a small class of curves
- One solution is to raise the degree
  - Allows more control, at the expense of more control points and higher degree polynomials
  - Control is not *local*, one control point influences entire curve
- Alternate, most common solution is to join pieces of cubic curve together into *piecewise cubic curves*
  - Total curve can be broken into pieces, each of which is cubic
  - *Local control*: Each control point only influences a limited part of the curve
  - Interaction and design is much easier

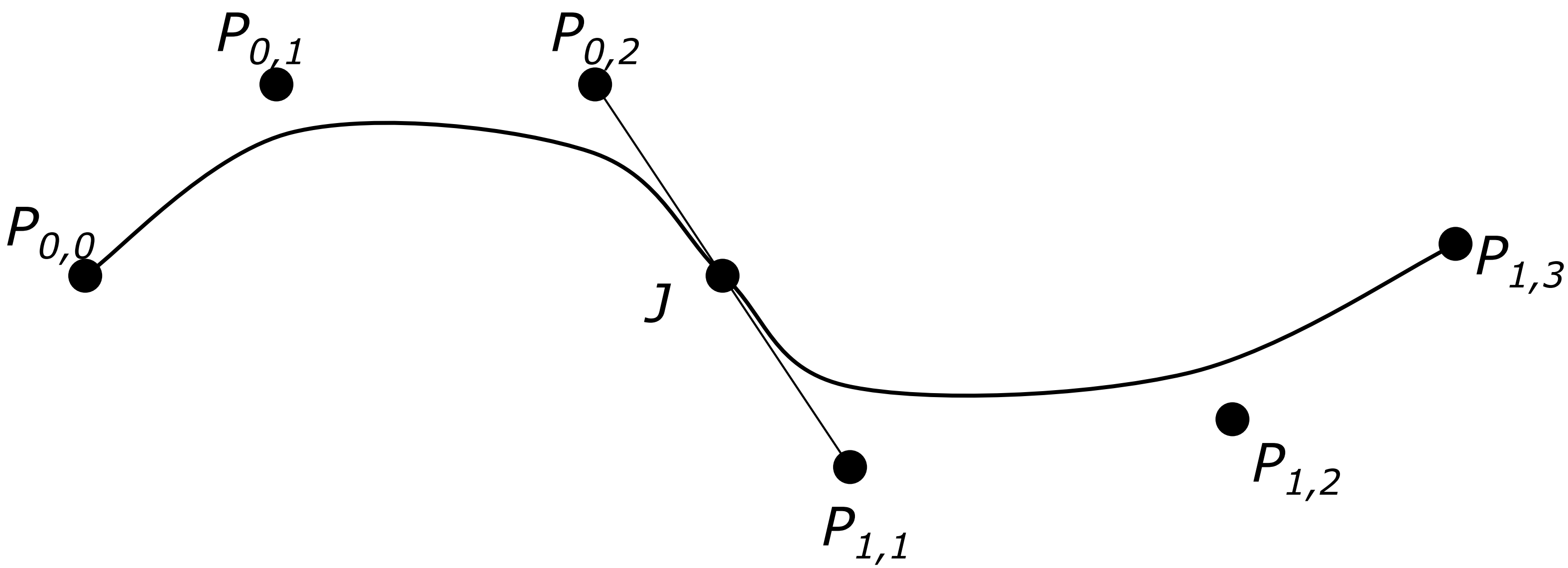
Piecewise Bezier Curve



Continuity

- When two curves are joined, we typically want some degree of continuity across the boundary (the knot)
  - C^0, "C-zero", point-wise continuous, curves share the same point where they join
  - C^1, "C-one", continuous derivatives, curves share the same parametric derivatives where they join
  - C^2, "C-two", continuous second derivatives, curves share the same parametric second derivatives where they join
  - Higher orders possible

Bezier Continuity



Disclaimer: PowerPoint curves are not Bezier curves, they are interpolating piecewise quadratic curves! This diagram is an approximation.

Sketch of Proof for C^1

Bezier curve equation:

$$\mathbf{x} = \mathbf{x}_0(1-t)^3 + \mathbf{x}_1 3t(1-t)^2 + \mathbf{x}_2 3t^2(1-t) + \mathbf{x}_3 t^3$$

$$= \mathbf{x}_0(1-3t+3t^2-t^3) + \mathbf{x}_1 3(t-2t^2+t^3) + \mathbf{x}_2 3(t^2-t^3) + \mathbf{x}_3 t^3$$

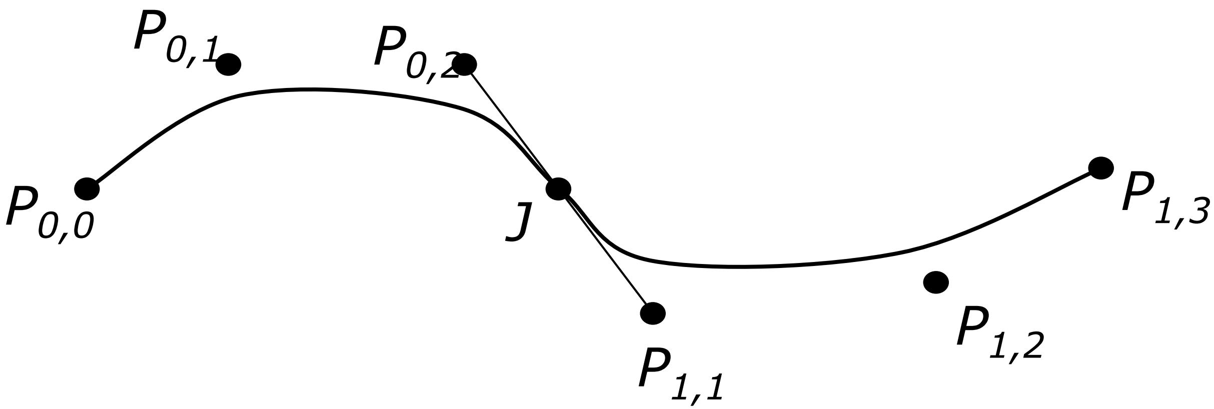
Parametric derivative:

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_0(-3+6t-3t^2) + \mathbf{x}_1 3(1-4t+3t^2) + \mathbf{x}_2 3(2t-3t^2) + \mathbf{x}_3 3t^2$$

Evaluated at endpoint of curve (note proves tangent property):

$$\left.\frac{d\mathbf{x}}{dt}\right|_{t=0} = -3\mathbf{x}_0 + 3\mathbf{x}_1 = 3(\mathbf{x}_1 - \mathbf{x}_0) \qquad \left.\frac{d\mathbf{x}}{dt}\right|_{t=1} = -3\mathbf{x}_2 + 3\mathbf{x}_3 = 3(\mathbf{x}_3 - \mathbf{x}_2)$$

Proof (cont)



C^1 requires equal parametric derivatives:

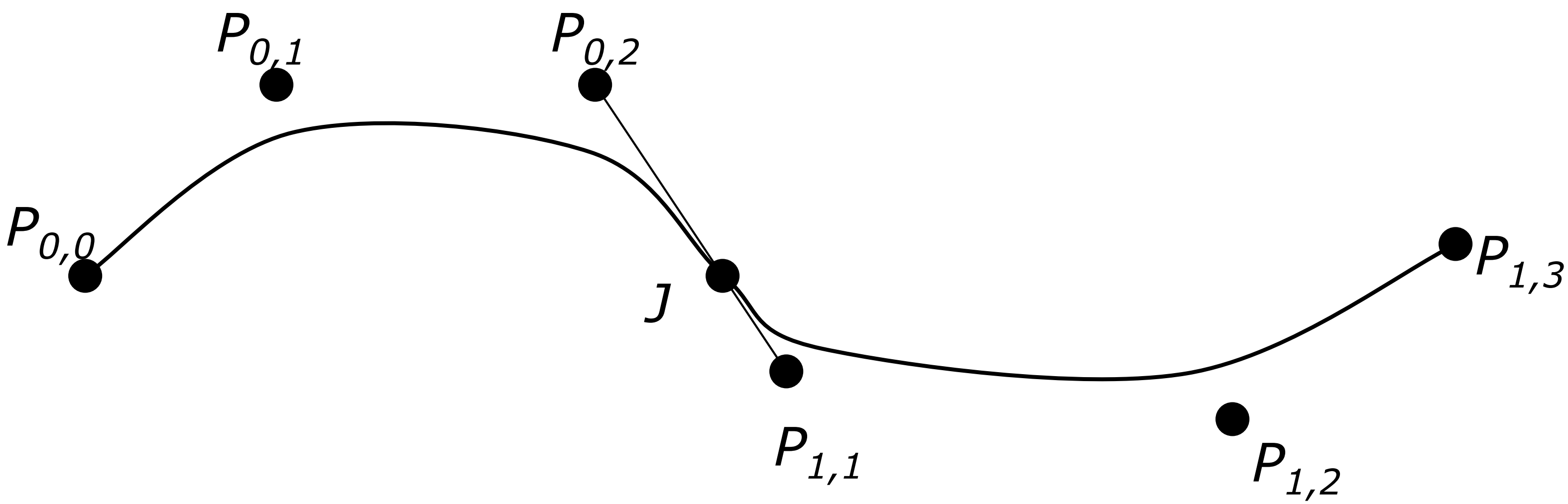
$$3(\mathbf{P}_{1,1} - \mathbf{J}) = 3(\mathbf{J} - \mathbf{P}_{0,2})$$

$$\mathbf{P}_{1,1} - \mathbf{J} = \mathbf{J} - \mathbf{P}_{0,2}$$

Geometric Continuity

- Derivative continuity is important for animation
  - If an object moves along the curve with constant parametric speed, there should be no sudden jump at the knots
- For other applications, *tangent continuity* might be enough
  - Requires that the tangents point in the same direction
  - Referred to as G^1 *geometric continuity*
  - Curves *could* be made C^1 with a re-parameterization: $u=f(t)$
  - The geometric version of C^2 is G^2, based on curves having the same radius of curvature across the knot
- What is the tangent continuity constraint for a Bezier curve?

Bezier Geometric Continuity



$$(\mathbf{P}_{1,1} - \mathbf{J}) = k(\mathbf{J} - \mathbf{P}_{0,2}) \quad \text{for some } k$$

Bezier Curve/Surface Problems

- To make a long continuous curve with Bezier segments requires using many segments
  - Same for large surface
- Maintaining continuity requires constraints on the control point positions
  - The user cannot arbitrarily move control vertices and automatically maintain continuity
  - The constraints must be explicitly maintained
  - It is not intuitive to have control points that are not free

B-splines

- B-splines automatically take care of continuity, with exactly one control vertex per curve segment
- Many types of B-splines: degree may be different (linear, quadratic, cubic,…) and they may be uniform or non-uniform
  - We will only look closely at uniform B-splines
- With uniform B-splines, continuity is always one degree lower than the degree of each curve piece
  - Linear B-splines have C^0 continuity, cubic have C^2, etc

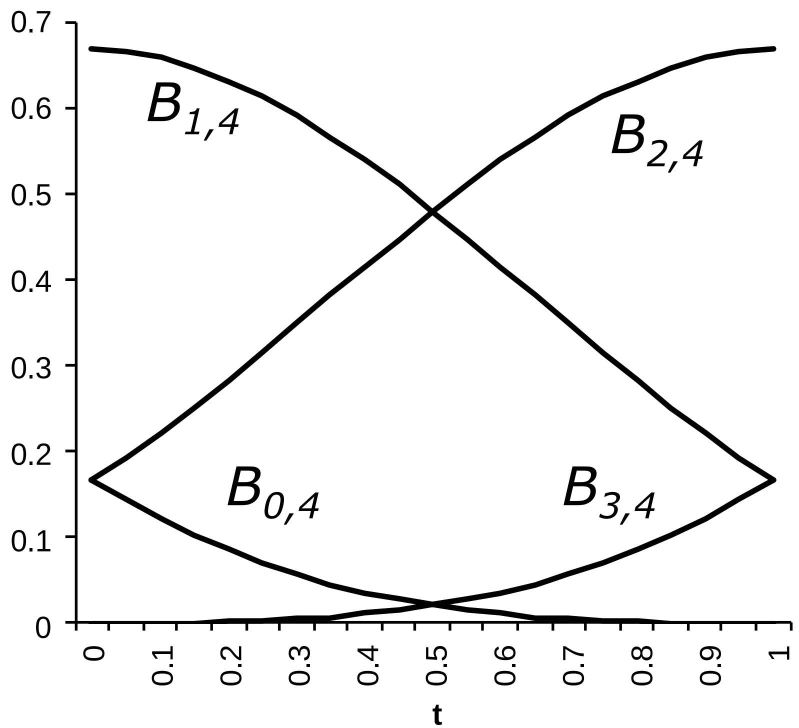
Uniform Cubic B-spline on [0,1)

- Four control points are required to define the curve for *0≤t<1* (*t* is the parameter)
  - Not surprising for a cubic curve with 4 degrees of freedom
- The equation looks just like a Bezier curve, but with different basis functions
  - Also called *blending functions* - they describe how to blend the control points to make the curve (see Shirley book ch. 15.6)

$$x(t) = \sum_{i=0}^{3} P_i B_{i,4}(t)$$

$$= P_0 \frac{1}{6}\left(1 - 3t + 3t^2 - t^3\right) + P_1 \frac{1}{6}\left(4 - 6t^2 + 3t^3\right) + P_2 \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right) + P_3 \frac{1}{6}\left(t^3\right)$$

Basis Functions on [0,1)



- Does the curve interpolate its endpoints?
- Does it lie inside its convex hull?

$$
\begin{aligned}
x(t) = {} & P_0 \frac{1}{6}\left(1 - 3t + 3t^2 - t^3\right) \\
& + P_1 \frac{1}{6_1}\left(4 - 6t^2 + 3t^3\right) \\
& + P_2 \frac{1}{6}\left(1 + 3t + 3t^2 - 3t^3\right) \\
& + P_3 \frac{1}{6}\left(t^3\right)
\end{aligned}
$$

Uniform Cubic B-spline on [0,1)

- The blending functions sum to one, and are positive everywhere
  - The curve lies inside its convex hull
- The curve does not interpolate its endpoints
  - Requires hacks or non-uniform B-splines
- There is also a matrix form for the curve:

$$x(t) = \frac{1}{6}\begin{bmatrix} P_0 & P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

Demo

Bspline-open.exe

Uniform B-spline at Arbitrary t

- The interval from an *integer* parameter value *v* to *v+1* is essentially the same as the interval from 0 to 1
  - The parameter value is offset by *v*
  - A different set of control points is needed
- To evaluate a uniform cubic B-spline at an arbitrary parameter value t:
  - Find the greatest integer less than or equal to *t*: *v* = floor(*t*)
  - Evaluate:

$$X(t) = \sum_{i=0}^{3} P_{i+v} B_{i,4}(t-v)$$

- Valid parameter range: $0 \le t < n-3$, where n is the number of control points

Loops

- To create a loop, use control points from the start of the curve when computing values at the end of the curve:

$$X(t) = \sum_{i=0}^{3} P_{(i+v) \bmod n} B_{i,4}(t-v)$$

- Any parameter value is now valid
  - Although for numerical reasons it is sensible to keep it within a small multiple of *n*

Demo

Bspline-closed.exe

B-splines and Interpolation, Continuity

- Uniform B-splines do not interpolate control points, unless:
  - You repeat a control point three times
  - But then all derivatives also vanish (=0) at that point
- To align tangents, use double control vertices
  - Then tangent aligns similar to Bezier curve
- Uniform B-splines are automatically C^2

How to Choose a Spline

- Hermite curves are good for single segments where you know the parametric derivative or want easy control of it
- Bezier curves are good for single segments or patches where a user controls the points
- B-splines are good for large continuous curves and surfaces

Next Time

- Raytracing