Contemporary Communication Systems

Chapter 2

Review of Signals and Linear Systems

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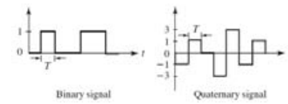
Analog and Digital Signals

- A continuous-time signal that assumes a continuum of amplitude values between given maximum and minimum is called an **analog** signal
  - Most signals we encounter in the real world are analog in nature. Examples include speech, music, image, and video signals
- **Digital signals**, on the other hand, can change values at discrete instants of time, assuming one of a finite number of amplitude levels

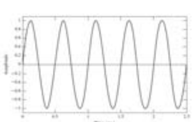
Analog signal

Digital signals

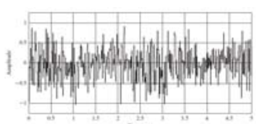


Deterministic and Random Signals

- A **deterministic** signal $x(t)$ is completely specified for each value of time t – that is, its amplitude is known either graphically or analytically for all values of t
  - An example is a simple sinusoidal waveform $\sin(4\pi t)$
- A **random** signal is not precisely known for each value of t – it can only be specified in terms of probabilities
  - A very important class of signals that includes noise and all information-carrying signals, such as speech and data



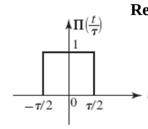
Sine wave



Random noise

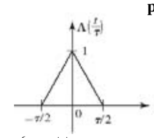
Some Useful Basic Signals

Rectangular pulse



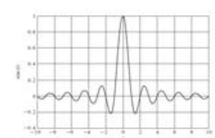
$$\Pi\left(\frac{t}{\tau}\right) \triangleq \begin{cases} 1 & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

Triangular pulse



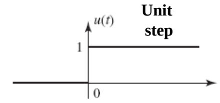
$$\Lambda\left(\frac{t}{\tau}\right) \triangleq \begin{cases} 1-\dfrac{2|t|}{\tau}, & |t| \le \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

Sinc pulse



$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

Unit step



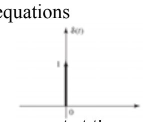
$$u(t) \triangleq \begin{cases} 1 & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The Unit Impulse Signal (Delta Function)

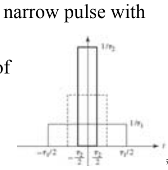
- The unit impulse signal $\delta(t)$ is defined by the equations

$$\delta(t) = 0, \quad t \ne 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1, \text{ for any real number } \varepsilon > 0$$



- Thus the unit impulse signal is zero everywhere except at the origin and it has unit area
- The value of $\delta(t)$ at $t = 0$ is not defined. In particular, $\delta(0) \ne \infty$
- A unit impulse signal can be viewed as a narrow pulse with large amplitude and having a unit area
- For example, it can be viewed as a limit of the unity area rectangular pulse as its width approaches zero and its amplitude increases proportionately



Important Properties of Delta Function

- **P1.** $x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$
- **P2.** $\delta(\alpha t) = \dfrac{1}{|\alpha|}\delta(t)$
- **P3. Sampling property**

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_o)dt = \int_{-\infty}^{\infty} x(t_o)\delta(t-t_o)dt = x(t_o)\int_{-\infty}^{\infty}\delta(t-t_o)dt = x(t_o)$$

- **P4. Convolution**

$$x(t)\otimes\delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

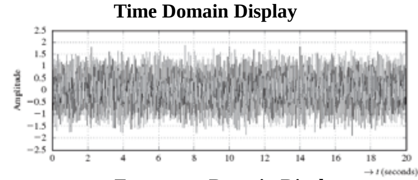
  - The convolution of an arbitrary signal with the impulse signal yields the signal itself

Frequency Domain Representation

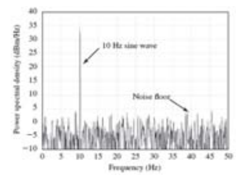
- Although electrical signals used in communication systems are functions of time, such as voltage and current, it is very useful to think of signals in terms of their frequency content
- Certain characteristics of signals are easier to analyze and measure in the frequency domain. In addition, the frequency domain analysis of many important operations on signals leads to unique and valuable insight towards understanding their effect
- That is why the frequency domain representation and analysis of signals and systems is an integral part of design tools for communication and control systems
- Figure shows the time domain representation of a 10 Hz sine wave embedded in noise

Sine Wave Embedded in Noise

Time Domain Display



Frequency Domain Display



Frequency Domain Representation (contd)

- Difficult to identify a 10 Hz tone in the presence of wideband ("white") noise on an oscilloscope display
- However, easy to identify 10 Hz tone in the frequency domain using a spectrum analyzer display. Note that the white noise forms the floor of the display
- In more complex situations, the composite signal may consist of hundreds of channels or carriers. An example is CATV system where several hundred channels or signals may be present
- Analyzing such a complex signal in time domain is not very useful. The frequency domain analysis, on the other hand, provides valuable insight into the effects of system impairments and noise when dealing with such signals

Exponential Fourier Series (FS)

- The Fourier series can be used to represent **periodic** signals in the frequency domain
- A periodic function $x_p(t)$ with fundamental period T_o can be represented by an **exponential** Fourier series

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}$$

where $f_o = 1/\,T_o$ is called the **fundamental** frequency of the periodic signal $x_p(t)$

- The FS coefficients C_n are given by

$$C_n = \frac{1}{T_o}\int_{T_o} x_p(t) e^{-j2\pi n f_o t} dt$$

- Observe that the FS expands a periodic function as an infinite sum of complex phasor signals

Exponential FS (contd)

- The term C_0 represents the **DC** component of the signal

$$C_0 = \frac{1}{T_o}\int_{T_o} x_p(t) dt$$

- $n = \pm 1$ FS coefficients represent the fundamental frequency (f_o) component in the periodic signal $x_p(t)$
- $n = \pm 2, \pm 3, \ldots$ FS coefficients represent the harmonic (nf_o) components in the periodic signal $x_p(t)$
- Each phasor term in FS can be written as

$$C_n e^{j2\pi n f_o t} = |C_n| e^{j(2\pi n f_o t + \angle C_n)}$$

- Plots of $|C_n|$ and $\angle C_n$ versus *discrete* frequency values ($n f_o$, $n = 0, \pm 1, \pm 2, \ldots$..) are called the **magnitude** and the **phase line spectra** of the signal, respectively

Trigonometric Fourier Series

- For a real signal,

$$C_{-n} = \frac{1}{T_o}\int_{T_0} x_p(t) e^{j2\pi n f_o t} dt = \left(\frac{1}{T_o}\int_{T_0} x_p(t) e^{-j2\pi n f_o t} dt\right)^* = C_n^*$$

- $\Rightarrow$ the magnitude spectrum is an even function, and the phase spectrum is an odd function of frequency
- In this case, the exponential FS can be expressed as

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t} = C_o + \sum_{n=1}^{\infty} C_{-n} e^{-j2\pi n f_o t} + \sum_{n=1}^{\infty} C_n e^{j2\pi n f_o t}$$

$$= C_0 + \sum_{n=1}^{\infty} |C_n| [e^{j(2\pi n f_o t + \angle C_n)} + e^{-j(2\pi n f_o t + \angle C_n)}]$$

$$= C_0 + 2\sum_{n=1}^{\infty} |C_n| \cos(2\pi n f_o t + \angle C_n)$$

Trigonometric Fourier series

Trigonometric FS (contd)

- An alternative form of the trigonometric FS is

$$x_p(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_o t) + \sum_{n=1}^{\infty} B_n \sin(2\pi n f_o t)$$

where

$$A_n = 2|C_n|\cos(\angle C_n) = C_n + C_n^* = \frac{2}{T_o}\int_{T_o} x_p(t)\cos(2\pi n f_o t)dt$$

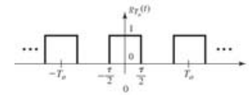
$$B_n = -2|C_n|\sin(\angle C_n) = -(C_n - C_n^*)/j = \frac{2}{T_o}\int_{T_o} x_p(t)\sin(2\pi n f_o t)dt$$

- $\Rightarrow$ For $x_p(t)$ even function of time, its FS will contain only cosine terms, i.e., $B_n = 0$, $n = 1, 2, \ldots\ldots$.
- $\Rightarrow$ For $x_p(t)$ odd function of time, its FS will contain only sine terms, i.e., $A_n = 0$, $n = 1, 2, \ldots\ldots$

Example: Rectangular Pulse Train

- Determine the FS expansion of a periodic pulse train of rectangular pulses

$$g_{T_o}(t) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{(t - nT_o)}{\tau}\right]$$

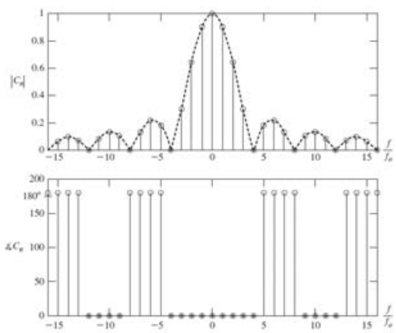


- Each pulse has unity amplitude and duration τ. The FS coefficients are given by

$$C_n = \frac{1}{T_o}\int_{T_o} g_{T_o}(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_0}\int_{-T_o/2}^{T_o/2} e^{-j2\pi n f_o t}dt = -\frac{1}{j2\pi n f_o T_o}\left[e^{-j2\pi n f_o \tau} - e^{j2\pi n f_o \tau}\right]$$

$$= \frac{\tau}{T_o}\frac{\sin(\pi n f_o \tau)}{\pi n f_o \tau} = \frac{\tau}{T_o}\text{sinc}(n f_o \tau)$$

Line Spectrum of a Rectangular Pulse Train



Energy and Power Signals

- Energy and power are useful parameters of a signal
- The normalized **energy** of a signal $x(t)$ is defined as the energy dissipated by a voltage $x(t)$ applied across a 1-ohm resistor (or a current $x(t)$ passing though a 1-ohm resistor)

$$E_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- The energy of a signal is meaningful only if the integral value is finite. Such signals are called **energy** signals
- **Example** Energy of a rectangular pulse $x(t) = A\Pi(t/T_b)$

$$x(t) = \begin{cases} A, & |t| \le T_b/2 \\ 0, & \text{otherwise} \end{cases}$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-T_b/2}^{T_b/2} A^2 dt = A^2 T_b$$

Example: Energy of the Carrier Pulse

$$x(t) = \begin{cases} A\cos(2\pi f_c t) & |t| \le T_b/2 \\ 0 & \text{otherwise} \end{cases}$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = A^2 \int_{-T_b/2}^{T_b/2} \cos^2(2\pi f_c t)dt = \frac{A^2}{2}\int_{-T_b/2}^{T_b/2} [1 + \cos(4\pi f_c t)]dt$$

$$= \frac{A^2}{2}\int_{-T_b/2}^{T_b/2} dt + \frac{A^2}{2}\int_{-T_b/2}^{T_b/2} \cos(4\pi f_c t)dt = \frac{A^2 T_b}{2}$$

- The second integral is zero because carrier frequency $f_c >> 1/T_b$ has been assumed – true in practice
- The energy content of the signal becomes infinite in the limit as $T_b \to \infty$

Power Signals

- The normalized average **power** of a signal $x(t)$ is defined as the power dissipated by a voltage $x(t)$ applied across a 1-ohm resistor (or a current $x(t)$ passing though a 1-ohm resistor)

$$P_x \triangleq \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

- The normalized average power of a signal is meaningful only if the limit exists (that is, finite). Such signals are called **power** signals
- For a periodic signal $x_p(t)$ with period T_o, the expression for normalized power simplifies to

$$P \triangleq \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x_p(t)|^2 dt$$

Example: Power of a Sinusoidal Signal

$$x_p(t) = A\cos(2\pi f_o t + \phi)$$

$$P = \frac{A^2}{T_o}\int_{-T_o/2}^{T_o/2} \cos^2(2\pi f_o t + \phi)dt$$

$$= \frac{A^2}{2T_o}[\int_{-T_o/2}^{T_o/2} dt + \int_{-T_o/2}^{T_o/2} \cos(4\pi f_c t + 2\phi)dt] = \frac{A^2}{2} + 0 = \frac{A^2}{2}$$

- The second integral is zero because it evaluates the integrand over two complete periods
- A signal cannot be both power- and energy-type, because for energy signals $P_x = 0$ and $E_x = \infty$ for power signals
  - A signal may be neither energy-type nor power-type

Parseval's Theorem

- The normalized power P_x of a periodic signal $x_p(t)$ is given by

$$P_x = \frac{1}{T_o}\int_{T_o} |x_p(t)|^2 dt = \frac{1}{T_o}\int_{T_o} x_p(t)x_p^*(t)dt$$

- Substituting the FS expansion for $x_p(t)$ yields

$$P_x = \frac{1}{T_o}\int_{T_o} x_p(t)\left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j2\pi n f_o t}\right]dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T_o}\int_{T_o} x_p(t)e^{-j2\pi n f_o t}dt\right]C_n^*$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Average power in the frequency component at $f = nf_o$

Average power of $x_p(t)$ = sum of the average power of phasor components

Bandwidth of a Signal

- The **bandwidth** of a signal is a measure of the frequency range that contains **significant** energy of the signal
- The term significant here implies inclusion of those frequencies that represent the signal with acceptable distortion
  - The latter is determined by the relevance in a given application.
- If the significant energy of the signal lies in the range of frequencies $f_1 < f < f_2$, the bandwidth would be $f_2 - f_1$
- There are many definitions of bandwidth depending on how frequencies f_1 and f_2 are chosen
- For example, if the frequencies f_1 and f_2 are chosen so that 99% of the power resides in the frequency band $f_1 < f < f_2$, the quantity $f_2 - f_1$ is called the **99% power bandwidth**

99% Power Bandwidth Rectangular Pulse Train

- We will assume $T_o = 1$ μsec and $\tau / T_o = 0.5$

$$f_o = \frac{1}{T_o} = 1 \text{ MHz}$$

- The FS coefficients of a rectangular pulse train are given by

$$C_n = \frac{A\tau}{T_o}\text{sinc}(nf_o\tau) = 0.5 \text{ sinc}(0.5n)$$

- The normalized average power P is

$$P = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} |x_p(t)|^2 dt = \frac{1}{T_o}\int_{-\tau/2}^{\tau/2} A^2 dt = \frac{A^2\tau}{T_o} = 0.5$$

- The power in various frequency components is given by

$$|C_n|^2 = 0.25 \ |\text{sinc}(0.5n)|^2$$

Example: Power in Various Frequency Components

- For example, power at DC frequency

$$|C_o|^2 = 0.25 \ |\text{sinc}(0)|^2 = 0.25$$

- As the table shows, we need to include 21 FS coefficients to get 99% power in the signal
- Since each spectral component is separated by 1 MHz, the 99% power bandwidth of the periodic pulse train is ~21 MHz.

n	C_n	Accumulated Power up to and including $f = nf_o$
0	0.5	0.25
1	0.6366	0.4526
3	–0.212	0.4752
5	0.1273	0.4833
7	–0.091	0.4874
9	0.0707	0.4899
11	–0.058	0.4916
13	0.0490	0.4928
15	–0.0424	0.4937
17	0.0374	0.4944
19	-0.0335	0.4949
21	0.0303	0.4954

Fourier Transform

- Any continuous-time signal $x(t)$ that has finite "energy", i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

can be represented in the frequency domain via the **Fourier transform (FT)**

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

- In general, $X(f)$ is a complex function of frequency f and can be written as

$$X(f) = |X(f)|e^{j\angle X(f)}$$

where $|X(f)|$ and $\angle X(f)$ are, respectively, called the **magnitude** and the **phase spectrum** of the signal $x(t)$

Fourier Transform (contd)

- The signal $x(t)$ can be recovered from its FT $X(f)$ using the **inverse Fourier transform** formula

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- Note that $X(f)$ is a continuous spectrum vs the line spectrum produced by FS coefficients C_n for a periodic signal
- Notation

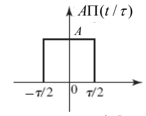
$$x(t) \xrightarrow{\Im} X(f) \text{ or } X(f) = \Im\{x(t)\} \quad \text{FT operation}$$

$$X(f) \xrightarrow{\Im^{-1}} x(t) \text{ or } x(t) = \Im^{-1}\{X(f)\} \quad \text{Inverse FT operation}$$

$$x(t) \xleftrightarrow{\Im} X(f) \quad \text{FT or Inverse FT operation}$$

FT of Rectangular Pulse

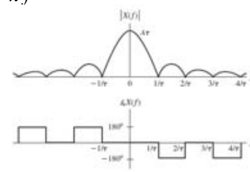
$x(t) = A\Pi(t/\tau)$



$$X(f) = A\int_{-\infty}^{\infty} \Pi(t/\tau) e^{-j2\pi ft} dt = A\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft} dt = A\frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{-j2\pi f}$$

$$= A\frac{\sin(\pi f\tau)}{\pi f} = A\tau \text{sinc}(f\tau)$$

Observe that the width of the mainlobe of $|X(f)|$ increases as the pulse width τ narrows



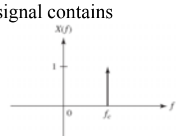
FT of Complex Exponential Signal $e^{j2\pi f_c t}$

$$e^{j2\pi f_c t} \xleftrightarrow{\Im} \delta(f - f_c) \qquad (*)$$

- This can be verified by substituting in the inverse Fourier transform formula as follows:

$$\int_{-\infty}^{\infty} \delta(f - f_c) e^{j2\pi ft} df = \int_{-\infty}^{\infty} \delta(f - f_c) e^{j2\pi f_c t} df = e^{j2\pi f_c t}$$

- The spectrum of a complex exponential signal contains energy at only single frequency f_c



- Substituting $f_c = 0$ into (*), we obtain the FT of a DC signal as

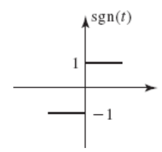
$$1 \xleftrightarrow{\Im} \delta(f)$$

FT of Signum Signal sgn(t)

- The signun signal sgn(t) can be expressed as

$$\text{sgn}(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t \leq 0 \end{cases}$$

$$= \lim_{\alpha \to 0} \begin{cases} e^{-\alpha t}, & t \geq 0 \\ e^{\alpha t}, & t \leq 0 \end{cases}$$



- The FT of sgn(t) is given by

$$\Im\{\text{sgn}(t)\} = \lim_{\alpha \to 0}\left[\int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi ft} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt\right]$$

$$= \lim_{\alpha \to 0}\left[\int_{-\infty}^{0} e^{(\alpha - j2\pi f)t} dt + \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t} dt\right]$$

$$= \lim_{\alpha \to 0} \frac{-4j\pi f}{\alpha^2 + 4\pi^2 f^2} = \frac{1}{j\pi f}$$

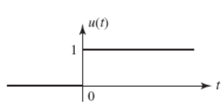
FT of Unit Step Signal

- The unit step function $u(t)$ can be expressed as

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

- Taking the FT of both sides yields

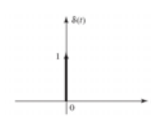
$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$



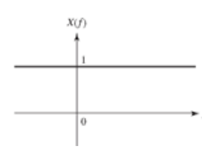
FT of Unit Impulse Signal

$x(t) = \delta(t)$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$



- $\Rightarrow$ the unit impulse signal contains all frequencies with equal magnitudes as shown in the Figure



Basic Fourier Transform Pairs

Time Function $x(t)$	Fourier Transform $X(f)$
DC signal A	$A\delta(f)$
Rectangular pulse $\Pi(t/\tau)$	$\tau\,\mathrm{sinc}(f\tau)$
Triangular pulse $\Lambda(t/\tau)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{f\tau}{2}\right)$
Decaying exponential $e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j2\pi f}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
Unit impulse $\delta(t)$	1
$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sinc pulse $\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\Pi(f/2W)$
Complex sinusoid $e^{j2\pi f_c t}$	$\delta(f - f_c)$
Sinusoid $\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
Sinusoid $\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
Gaussian pulse $e^{-\pi t^2}$	$e^{-\pi f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
Unit step $u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\frac{1}{\pi t}$	$-j\,\mathrm{sgn}(f)$
$\delta'(t)$	$j2\pi f$

Properties of Fourier Transform

- There are a number of important properties of the Fourier transform which are useful in the analysis and design of communication and control systems
- **Linearity** $x(t) \stackrel{\Im}{\longleftrightarrow} X(f)$

  $y(t) \stackrel{\Im}{\longleftrightarrow} Y(f)$

  $$\alpha x(t) + \beta y(t) \stackrel{\Im}{\longleftrightarrow} \alpha X(f) + \beta Y(f)$$
- Taking the Fourier Transform of the left hand side yields

$$\alpha x(t) + \beta y(t) \stackrel{\Im}{\longleftrightarrow} \int_{-\infty}^{\infty}[\alpha x(t) + \beta y(t)]e^{-j2\pi ft}dt$$
$$= \alpha\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt + \beta\int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt$$
$$= \alpha X(f) + \beta Y(f)$$

Properties of FT: Conjugate Symmetry

- For real $x(t)$,

  $$X(-f) = \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt = X^*(f)$$
- Comparing magnitude and phase responses of both sides of yields

  $$|X(-f)| = |X(f)|$$

  $$\angle X(-f) = -\angle X(f)$$
- Thus $|X(f)|$ and $\angle X(f)$ are even and odd functions of f, respectively.

Properties of FT

- **Time Shifting:** $x(t - t_o) \stackrel{\Im}{\longleftrightarrow} X(f)e^{-j2\pi f t_o}$
- This can be proved by using the inverse FT formula.

$$x(t - t_o) = \int_{-\infty}^{\infty} X(f)e^{j2\pi f(t - t_o)}df = \int_{-\infty}^{\infty}[X(f)e^{-j2\pi f t_o}]e^{j2\pi ft}df = \Im^{-1}\{X(f)e^{-j2\pi f t_o}\}$$

- Note that the magnitude of the FT is unchanged by a time shift. However, it introduces a linear phase shift of $e^{-j2\pi f t_o}$
- **Frequency Translation:** $x(t)e^{j2\pi f_c t} \stackrel{\Im}{\longleftrightarrow} X(f - f_c)$
- Taking the Fourier transform of the left hand side yields

  $$\int_{-\infty}^{\infty} x(t)e^{j2\pi f_c t}e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f - f_c)t}dt = X(f - f_c)$$
- $\Rightarrow$ that multiplication of a signal $x(t)$ by $e^{j2\pi f_c t}$ translates its frequency spectrum $X(f)$ by the amount f_c (to the right)

Properties of FT: Convolution

$$x(t) \otimes y(t) \stackrel{\Im}{\longleftrightarrow} X(f)Y(f)$$

- From the definition of convolution operation

  $$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau$$
- If we take the Fourier transform of the right hand side, and exchange the order of integration, we get

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau\right]e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} d\tau\, x(\tau)\left[\int_{-\infty}^{\infty} y(t - \tau)e^{-j2\pi ft}dt\right]$$
$$= Y(f)\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = Y(f)X(f)$$

- $\Rightarrow$ the convolution operation in time-domain is **equivalent** to multiplication in the frequency domain

Properties of FT: Multiplication Property

- This property is the dual of the convolution property. The multiplication of two signals results in the convolution of their spectra

  $$x(t)y(t) \stackrel{\Im}{\longleftrightarrow} X(f) \otimes Y(f)$$
- Modulation involves multiplication of a signal $x(t)$ by a high-frequency sinusoidal waveform. That is,

  $y(t) = \cos(2\pi f_c t)x(t)$
- Applying the multiplication property

$$Y(f) = \left(\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)\right) \otimes X(f)$$
$$= \frac{1}{2}X(f - f_c) + \frac{1}{2}X(f + f_c)$$

Spectrum $X(f)$ shifted by carrier frequency f_c

Properties of FT: Time/Frequency Scaling

$$x(at) \overset{\Im}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- Assume $a > 0$. Using the transformation of variables, $u = at$, we have

$$x(at) \overset{\Im}{\longleftrightarrow} = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt = \frac{1}{a} \int_{-\infty}^{\infty} x(u) e^{-j2\pi (f/a) u} du = \frac{1}{a} X\left(\frac{f}{a}\right)$$

- Now with $a < 0$, substituting $u = -|a|t$ yields

$$x(at) \overset{\Im}{\longleftrightarrow} = \int_{-\infty}^{\infty} x(-|a|t) e^{-j2\pi ft} dt = \frac{1}{|a|} \int_{-\infty}^{\infty} x(u) e^{j2\pi (f/|a|) u} du$$

$$= \frac{1}{|a|} X\left(-\frac{f}{a}\right) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

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Properties of Fourier Transform (contd)

- The function $x(at)$, for $a > 0$, is a time compressed (by a factor a) version of $x(t)$. On the other hand, a function $X(f/a)$ represents a function $X(f)$ expanded by the same factor a.
- The scaling property therefore states that compressing a signal in time domain will stretch its Fourier transform. Similarly stretching a time signal will compress its Fourier transform.
- The result is intuitively satisfying since compression in time by the factor $a > 0$ means that the function is varying rapidly in time by the same amount
- Consequently, the frequencies of its components will be increased by the factor a. The converse can also be justified by a similar argument.

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Properties of FT: Duality

- If $x(t) \overset{\Im}{\longleftrightarrow} X(f)$, then $X(t) \overset{\Im}{\longleftrightarrow} x(-f)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- Making a change of variable $f = -v$ yields

$$x(t) = \int_{-\infty}^{\infty} X(-v) e^{-j2\pi vt} dv$$

- If we set $t = -f$, we get

$$x(-f) = \int_{-\infty}^{\infty} X(-v) e^{j2\pi fv} dv$$

- Finally, substituting t for $-v$, we get

$$x(-f) = \int_{-\infty}^{\infty} X(t) e^{-j2\pi ft} dt = \Im\{X(t)\}$$

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Example: Duality

- From FT Table

$$\Pi(t/2\tau) \overset{\Im}{\longleftrightarrow} 2\tau \mathrm{sinc}(2f\tau) \qquad \left(x(t) \overset{\Im}{\longleftrightarrow} X(f)\right)$$

- Using duality property of the FT

$$X(t) \overset{\Im}{\longleftrightarrow} x(-f)$$

$$2W \mathrm{sinc}(2tW) \overset{\Im}{\longleftrightarrow} \Pi(f/2W)$$

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Properties of FT: Differentiation Property

$$\frac{d}{dt} x(t) \overset{\Im}{\longleftrightarrow} j2\pi f X(f)$$

- To prove this, we have

$$\frac{d}{dt} x(t) = \frac{d}{dt} \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} X(f) \left(\frac{d}{dt} e^{j2\pi ft}\right) df$$

$$= \int_{-\infty}^{\infty} \left[j2\pi f X(f) \right] e^{j2\pi ft} df$$

- That is, differentiation in time domain is equivalent to multiplication by □$j2\pi f$ in the frequency domain

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Properties of FT: Parseval's Relation

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(f) Y^*(f) df$$

- To prove, substituting $y^*(t) = \int_{-\infty}^{\infty} Y^*(f) e^{-j2\pi ft} df$ and exchanging the order of integration yields

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y^*(f) e^{-j2\pi ft} df \right] dt$$

$$= \int_{-\infty}^{\infty} Y^*(f) \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \right] df = \int_{-\infty}^{\infty} Y^*(f) X(f) df$$

- We get well-known relationship for the energy of a signal in time and frequency domains by letting $y(t) = x(t)$ in Parseval's relation

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad (*)$$

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Energy Spectral Density

- Equation (*) states that the energy of a signal is given by the area under the curve $|X(f)|^2$
  - $|X(f)|^2$ is called the **energy spectral density** of $x(t)$
- Note that the quantity $|X(f_o)|^2 \Delta f$ represents the energy contained in 2 spectral bands of Δf Hz centered at frequencies $\pm f_o$
- Thus $|X(f)|^2$ may be interpreted as the energy contained in the spectral components of $x(t)$ centered at frequency f per Hz of bandwidth
- $|X(f)|^2$ thus represents distribution of energy as a function of frequency for a signal. So when integrated, it provides the energy of the signal
  - The energy spectral density of a signal is specified in units of Watts-sec/Hz

Fourier Transform Properties

Property	Time Function $x(t)$, $y(t)$	Fourier Transform $X(f)$, $Y(f)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(f) + \beta Y(f)$
Time-shifting	$x(t - t_o)$	$X(f)e^{-j2\pi f t_o}$
Frequency translation	$x(t)e^{j2\pi f_c t}$	$X(f - f_c)$
Convolution	$x(t) \otimes y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$X(f) \otimes Y(f)$
Time/Frequency scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$
Duality	$X(t)$	$x(-f)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j2\pi f X(f)$
Differentiation in frequency	$tx(t)$	$\frac{j}{2\pi}\frac{d}{df}X(f)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
Parseval's relation	$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$	

Fourier Transforms of Periodic Signals

- The Fourier transform is strictly defined for finite energy signals. However, we can extend its scope by allowing the FT to include delta functions
- Since a periodic signal can be expanded into exponential FS, its FT can be obtained by taking the FT of the FS term by term
- The FT expansion of a periodic function $x_p(t)$ is obtained as

$$X_p(f) = \Im\left\{\sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_o t}\right\} = \sum_{n=-\infty}^{\infty} C_n \Im\left\{e^{j2\pi n f_o t}\right\}$$

$$= \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_o)$$

- $\Rightarrow$ the FT of a periodic signal consists of impulses located at harmonic frequencies of the signal. The weight of each impulse equals FS coefficient, i.e., $X_p(nf_o) = C_n$

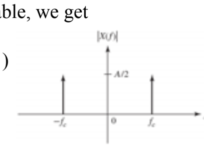
Fourier Transforms of Periodic Signals (contd)

- **Example: Fourier transform of a cosine wave**

$$x(t) = A\cos(2\pi f_c t + \phi) = \frac{A}{2}e^{j(2\pi f_c t + \phi)} + \frac{A}{2}e^{-j(2\pi f_c t + \phi)}$$

- Taking the FT of both sides and using Table, we get

$$X(f) = \frac{A}{2}e^{j\phi}\delta(f - f_c) + \frac{A}{2}e^{-j\phi}\delta(f + f_c)$$



- Similarly, it can be shown that

$$A\sin(2\pi f_c t + \varphi) \overset{\Im}{\longleftrightarrow} \frac{A}{2j}e^{j\varphi}\delta(f - f_c) - \frac{A}{2j}e^{-j\varphi}\delta(f + f_c)$$

FT of Periodic Impulse Train

- The periodic impulse train with period T_o is given by

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_o)$$

- The coefficient in its FS expansion is

$$C_n = \frac{1}{T_o}\int_{T_o} \delta_p(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} \delta(t)e^{-j2\pi n f_o t}dt = \frac{1}{T_o} = f_o$$

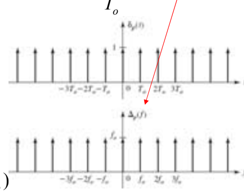
FT also a periodic impulse train

- The FS can now be expressed as

$$\delta_p(t) = f_o \sum_{n=-\infty}^{\infty} e^{j2\pi n f_o t}$$

- Taking the FT of both sides

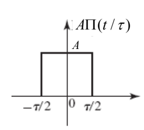
$$\Delta_p(f) = \Im\{\delta_p(t)\} = f_o \sum_{n=-\infty}^{\infty} \delta(f - nf_o)$$



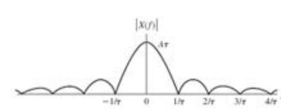
Time-bandwidth Product

- Recall the scaling property of the Fourier transform, which states that the compression in the time domain is equivalent to the expansion in the frequency domain, and vice versa
- Thus, the frequency- and time-domain behaviors of a signal are **inversely** related
  - $\Rightarrow$ a signal cannot be both time-limited and bandwidth-limited

Time-limited



Not Band-limited



Time-bandwidth Product (contd)

- However, a signal can be "approximately" time-limited and band-limited
  - That is, there exist numbers $B > 0$ and $T > 0$, such that $|x(t)|$ is small for $|t| \geq T$ and $|X(f)|$ is small for $|f| \geq B$
- The product of a signal's duration and its bandwidth is constant.
  - **Duration × Frequency Bandwidth = *k***
- The constant k is determined by the precise definitions of **duration** in the time domain and **bandwidth** in the frequency domain
  - For a Gaussian pulse, if we use the RMS definitions of duration and bandwidth of a signal, it can be shown that

$$\Delta f_{rms} \Delta t_{rms} = \frac{1}{4\pi}$$

Transmission of Signals Through LTI Systems

- An **Linear Time-invariant (LTI)** system is completely characterized in the time domain by its impulse response $h(t)$

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

- Applying the FT to both sides and using convolution property

$$Y(f) = X(f) H(f)$$

- $\Rightarrow$ that the output of the system in the frequency domain is given by multiplying the Fourier transform of the input by the system frequency response $H(f)$

$x(t)$ / $X(f)$ → LTI system $h(t) \overset{\Im}{\longleftrightarrow} H(f)$ → $y(t) = x(t) \otimes h(t)$ / $Y(f) = X(f)H(f)$

Frequency Response of LTI Systems

- $H(f)$, in general, is a complex function of f. That is,

$$H(f) = |H(f)| e^{j\angle H(f)}$$

where $|H(f)|$ and $\angle H(f)$ are, respectively, called the **magnitude** and the **phase responses** of the system

- If $h(t)$ is a real function, applying conjugation property of FT

$$\underbrace{|H(f)| = |H(-f)|}_{\text{even function of } f}, \qquad \underbrace{\angle H(f) = -\angle H(-f)}_{\text{odd function of } f}$$

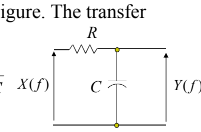
- In the frequency domain, the magnitude and the phase of the system input and output are related by

$$|Y(f)| = |X(f)| \times |H(f)|$$

$$\angle Y(f) = \angle X(f) + \angle H(f)$$

Example: First-order RC LP filter

- The first-order RC LP filter is shown in Figure. The transfer function of the network is given by

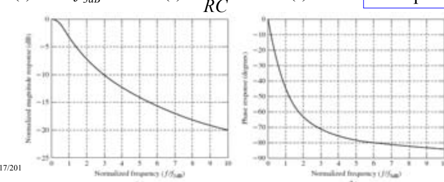


$$H(f) = \frac{Y(f)}{X(f)} = \frac{1/j2\pi fC}{R + 1/j2\pi fC} = \frac{1}{1 + j2\pi fRC}$$

$$= \frac{1}{1 + j(f/f_{3dB})}$$

3-dB cutoff frequency

$$h(t) = 2\pi f_{3dB} e^{-2\pi f_{3dB} t} u(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Distortionless Transmission

- An LTI system is termed **distortionless** if it introduces the same attenuation to all spectral components and offers linear phase response over the frequency band of interest, that is,

$$H_{ideal}(f) = \begin{cases} H_o e^{-j2\pi f t_o} & f_1 \leq f \leq f_2 \\ 0 & \text{otherwise} \end{cases}$$

- Substituting yields

$$Y(f) = X(f) H_{ideal}(f) = H_o X(f) e^{-j2\pi f t_o}$$

- Taking the FT of both sides, the output of a distortionless LTI system due to an arbitrary input signal $x(t)$ is given by

$$y(t) = H_o x(t - t_o)$$

delayed and scaled replica of the input

Group Delay

- The group delay of an LTI system is defined as

$$\tau_g(f) \triangleq -\frac{1}{2\pi} \frac{d\angle H(f)}{df}$$

  - Represents time delay incurred by a spectral component at frequency f as it passes through the LTI system
- The phase response of an ideal filter is linear function of frequency as given by

$$\angle H_{ideal}(f) = -2\pi f t_o, \quad f_1 \leq f \leq f_2, \; t_o = \text{constant}$$

- For a linear phase LTI system, we obtain

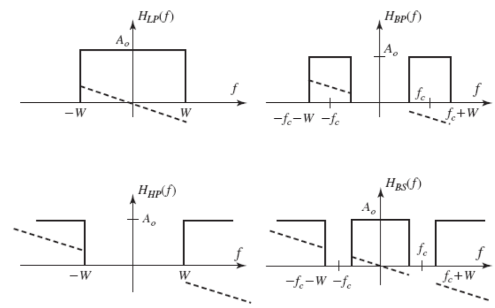
$$\tau_g(f) = -\frac{1}{2\pi} \frac{-2\pi f t_o}{df} = t_o = \text{constant}$$

- $\Rightarrow$ all frequency components of the input signal undergo the same time delay through the system $\Rightarrow$ *no distortion*

Ideal Filters

- One key application of LTI systems is processing of signals in order to enhance certain frequency components and to reject certain others
  - For example, if a signal consists of a low-frequency information-bearing message waveform and high-frequency noise, we can employ a filter to reject the high frequencies and thus remove the noise
- An ideal filter designed to pass certain frequency components should have a magnitude response that is constant and phase response that is linear over these frequencies– called the **passband** of the filter
- The magnitude response of the ideal filter is zero over the range of frequencies blocked by the filter – called the **stopband** of the filter

Ideal Filters (contd)



Ideal Lowpass Filter

- The magnitude response of an ideal **lowpass (LP)** filter is

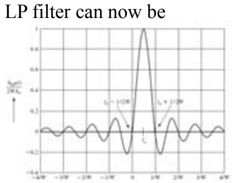
$$|H_{LP}(f)| = \begin{cases} A_o & -W \le f \le W \\ 0 & \text{otherwise} \end{cases}$$

- The passband of the filter is range of frequencies $0 \le f \le W$. The range of frequencies $f > W$ is the stopband of the filter
- The frequency response of an ideal LP filter can now be written as

$$H_{LP}(f) = A_o \Pi(f/2W) e^{-j2\pi f t_o}$$

- Taking the inverse FT yields

$$h_{LP}(t) = 2WA_o \text{sinc}[2W(t-t_o)]$$



Ideal Highpass Filter

- The magnitude response of an ideal **highpass (HP)** filter is

$$|H_{HP}(f)| = \begin{cases} 0 & -W \le f \le W \\ A_o & \text{otherwise} \end{cases}$$

- The range of frequencies $f \le W$ is the stopband of the filter. The range of frequencies $f > W$ is the passband of the filter
- The frequency response of an ideal HP filter can now be written as

$$H_{HP}(f) = A_o[1 - \Pi(f/2W)] e^{-j2\pi f t_o}$$

- Taking the inverse FT yields

$$H_{HP}(f) = A_o\left[1 - \Pi(f/2W)\right] e^{-j2\pi f t_o}$$

Ideal Bandpass Filter

- The amplitude response of an ideal **bandpass (BP)** filter is

$$|H_{BP}(f)| = \begin{cases} A_o & f_c - W \le |f| \le f_c + W \\ 0 & \text{otherwise} \end{cases}$$

- The range of frequencies $f_c - W \le |f| \le f_c + W$ is the passband of the filter. The range of frequencies $|f| > f_c + W$ and $|f| < f_c - W$ are the stopband regions of the filter
- The frequency response of an ideal BP filter can now be written as

$$H_{BP}(f) = H_o(f - f_c) + H_o(f + f_c)$$

where

$$H_o(f) = A_o \Pi(f/2W) e^{-j2\pi f t_o}$$

Ideal Bandpass Filter (contd)

- $H_o(f)$ is a LP filter with impulse response

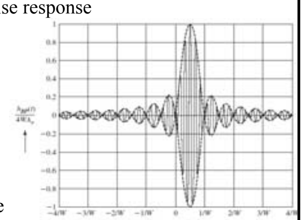
$$h_o(t) = 2WA_o \text{sinc}[2W(t-t_o)]$$

- Since

$$H_o(f - f_c) \overset{\Im}{\longleftrightarrow} h_o(t) e^{j2\pi f_c t}$$
$$H_o(f + f_c) \overset{\Im}{\longleftrightarrow} h_o(t) e^{-j2\pi f_c t}$$

we can now write the impulse response $h_{BP}(t)$ of the BP filter as

$$h_{BP}(t) = 4WA_o \text{sinc}\left[2W(t-t_o)\right]\left[\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right]$$
$$= 4WA_o \text{sinc}\left[2W(t-t_o)\right] \cos(2\pi f_c t)$$



Ideal Bandstop Filter

- The amplitude response of an ideal *bandstop* (BS) filter is defined as

$$|H_{BS}(f)| = \begin{cases} A_o & \text{otherwise} \\ 0 & f_c - W \le |f| \le f_c + W \end{cases}$$

- The range of frequencies $f_c - W \le |f| \le f_c + W$ is the stopband of the filter. The range of frequencies $|f| > f_c + W$ and $|f| < f_c - W$ are the passband regions of the filter.

Power Density Spectrum

- In the design of communication systems, we are interested in power distribution of a power signal in the frequency domain
- The problem in dealing with power signals in the frequency domain is that their Fourier transform may not exist as they have infinite energy
- To overcome this problem, we define a new function $x_T(t)$ by truncating $x(t)$ outside the interval $|t| > T/2$

$$x_T(t) = \begin{cases} x(t) & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

- $x_T(t)$ has finite energy as long as T is finite. Using Parseval's relation

$$E_{x_T} = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

Power Density Spectrum (contd)

- Since

$$\int_{-T/2}^{T/2} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x_T(t)|^2 dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

the average power of signal can be expressed as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

- Since $x(t)$ is a power signal, the integral on the right hand side exists in the limit as $T \to \infty$. Therefore, we can change the order of integration and limit yielding

$$P_x = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{T} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} df$$

Power Spectral Density

- The **power spectral density (PSD)** $\mathcal{G}_x(f)$ of power signal $x(t)$ is defined as

$$\mathcal{G}_x(f) \triangleq \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

- This allows us to express the normalized average power as

$$P_x = \int_{-\infty}^{\infty} \mathcal{G}_x(f) df$$

- Again $\mathcal{G}_x(f_o)\Delta f$ represents the power contained in 2 spectral bands of width Δf Hz centered at frequencies $\pm f_o$
- Thus $\mathcal{G}_x(f)$ may be interpreted as the power contained in spectral components of $x(t)$ centered at frequency f per Hz of bandwidth. It is specified in units of W/Hz.

PSD of Periodic Signals

- For a periodic signal $x_p(t)$, the normalized average power is given by

$$P_x = \sum_{n=-\infty}^{\infty} |C_n|^2$$

- Since $|C_n|^2$ is power contained in the spectral component at $f = nf_o$, the PSD $\mathcal{G}_x(f)$ of a periodic signal can be expressed as

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_o)$$

- Since FS coefficient $C_n = X_p(nf_o)$, we can express the PSD as

$$\mathcal{G}_x(f) = \sum_{n=-\infty}^{\infty} |X_p(nf_o)|^2 \delta(f - nf_o)$$

Time-average Autocorrelation Function

- The **time-average autocorrelation function** of a power signal $x(t)$ is defined as

$$\mathcal{R}_x(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

- The normalized average power P_x of $x(t)$ is related to $\mathcal{R}_x(\tau)$ by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \mathcal{R}_x(0)$$

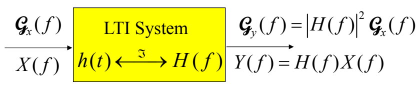
- It can be shown that the PSD of a power signal $x(t)$ is the FT of its time-average autocorrelation function

$$\mathcal{G}_x(f) \xleftrightarrow{\mathfrak{F}} \mathcal{R}_x(\tau)$$

Response of LTI System: Deterministic Inputs

- For a linear system with transfer function $H(f)$, the output $y(t)$ in response to a deterministic input signal $x(t)$ is given by

$$Y(f) = X(f)H(f)$$



- The PSD of a power signal $y(t)$ can be written as

$$\mathcal{G}_y(f) = \lim_{T\to\infty} \frac{|Y_T(f)|^2}{T}$$

- Now

$$Y_T(f) = X_T(f)H(f)$$

Response of LTI System (contd)

- Substituting yields

$$\mathcal{G}_y(f) = \lim_{T\to\infty} \frac{|X_T(f)H(f)|^2}{T} = |H(f)|^2 \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$
$$= |H(f)|^2 \mathcal{G}_x(f)$$

- $\Rightarrow$ that the output signal PSD in an LTI system depends on the magnitude of $H(f)$, and is given by $|H(f)|^2$ times the input PSD
- Taking inverse FT of both sides and applying the convolution property, we obtain

$$\mathcal{R}_y(\tau) = h(\tau) \otimes h^*(-\tau) \otimes \mathcal{R}_x(\tau)$$