5. Capacity of Wireless Channels

Information Theory

- So far we have only looked at specific communication schemes.
- Information theory provides a fundamental limit to (coded) performance.
- It succinctly identifies the impact of channel resources on performance as well as suggests new and cool ways to communicate over the wireless channel.
- It provides the basis for the modern development of wireless communication.

Capacity of AWGN Channel

Capacity of AWGN channel

$$\begin{aligned} C_{\text{awgn}} &= \log(1+\text{SNR}) \quad \text{bits/s/Hz} \\ &= W\log(1+\text{SNR}) \quad \text{bits/s} \end{aligned}$$

If average transmit power constraint is $\bar{P}$ watts and noise psd is N_0 watts/Hz,

$$C_{\text{awgn}} = W\log\left(1+\frac{\bar{P}}{N_0 W}\right) \quad \text{bits/s.}$$

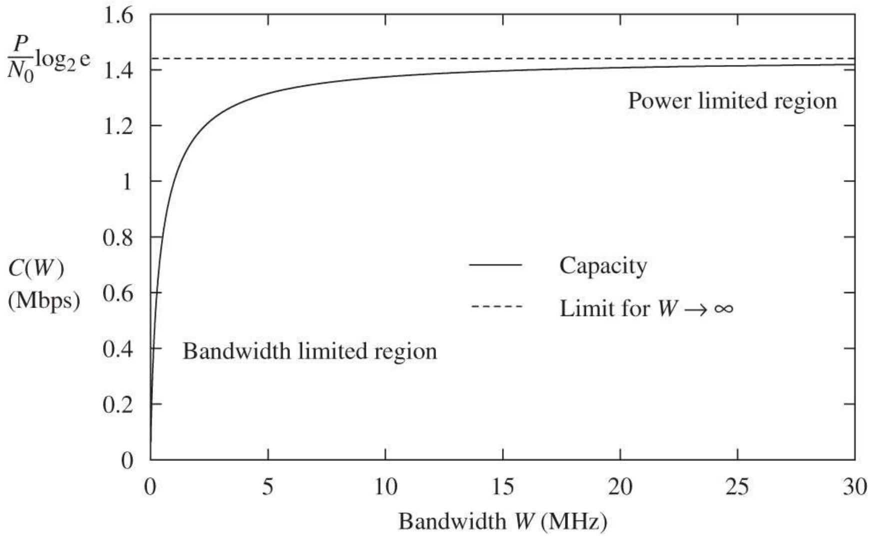
Power and Bandwidth Limited Regimes

$$C_{\text{awgn}} = W\log\left(1+\frac{\bar{P}}{N_0 W}\right)$$

$$\text{SNR} = \frac{\bar{P}}{N_0 W}$$

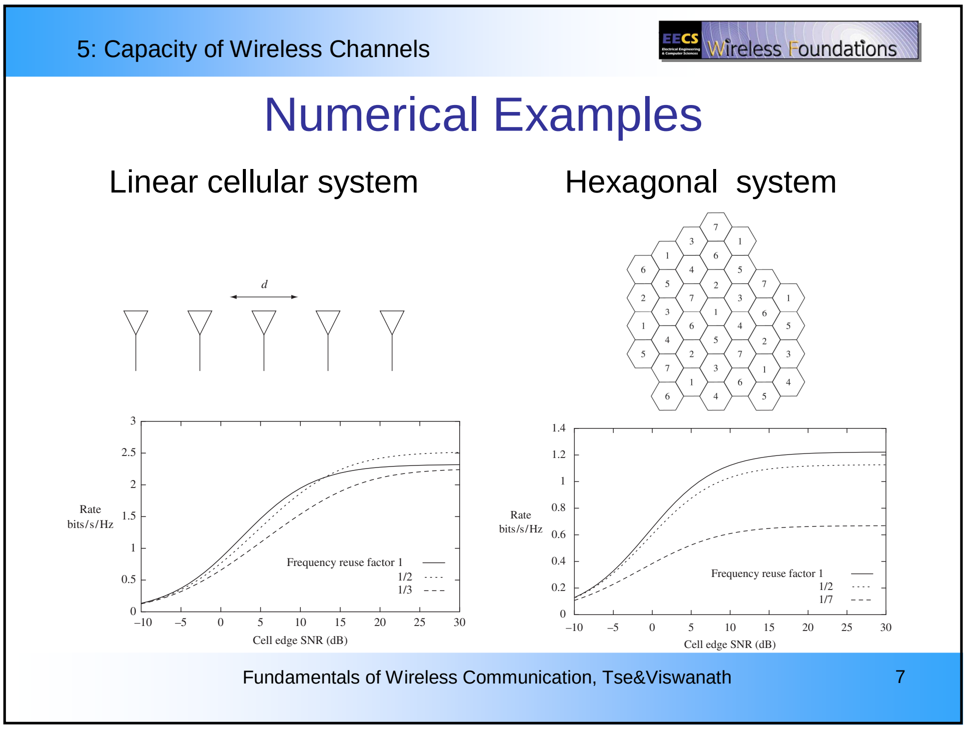
Bandwidth limited regime $\text{SNR} \gg 1$: capacity logarithmic in power, approximately linear in bandwidth.

Power limited regime $\text{SNR} \ll 1$: capacity linear in power, insensitive to bandwidth.



Example 1: Impact of Frequency Reuse

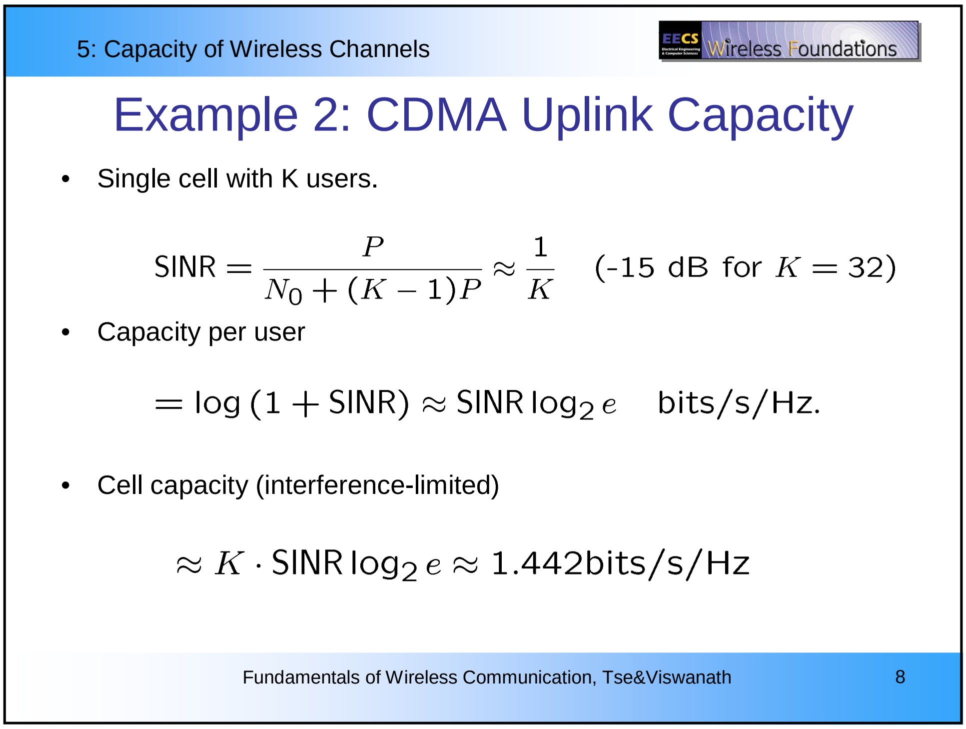
- Different degree of frequency reuse allows a tradeoff between SINR and degrees of freedom per user.
- Users in narrowband systems have high link SINR but small fraction of system bandwidth.
- Users in wideband systems have low link SINR but full system bandwidth.
- Capacity depends on both SINR and d.o.f. and can provide a guideline for optimal reuse.
- Optimal reuse depends on how the out-of-cell interference fraction f(ρ) depends on the reuse factor ρ.



Numerical Examples

Linear cellular system

Hexagonal system



Example 2: CDMA Uplink Capacity

- Single cell with K users.

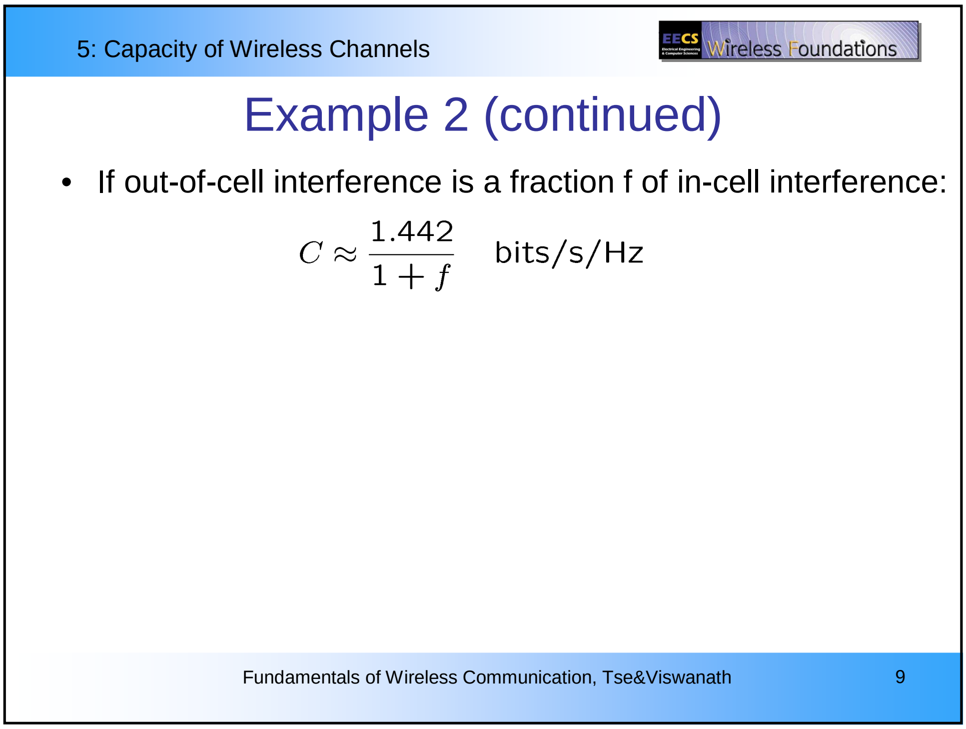
$$\mathrm{SINR} = \frac{P}{N_0 + (K-1)P} \approx \frac{1}{K} \quad \text{(-15 dB for } K = 32\text{)}$$

- Capacity per user

$$= \log(1 + \mathrm{SINR}) \approx \mathrm{SINR} \log_2 e \quad \text{bits/s/Hz.}$$

- Cell capacity (interference-limited)

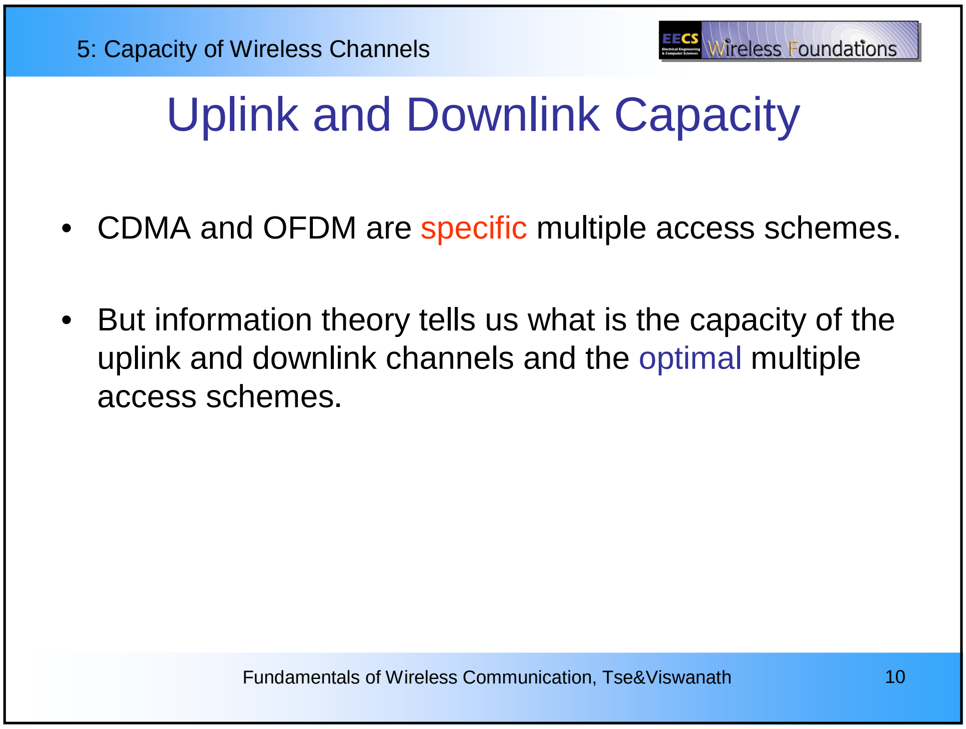
$$\approx K \cdot \mathrm{SINR} \log_2 e \approx 1.442 \text{bits/s/Hz}$$



Example 2 (continued)

- If out-of-cell interference is a fraction f of in-cell interference:

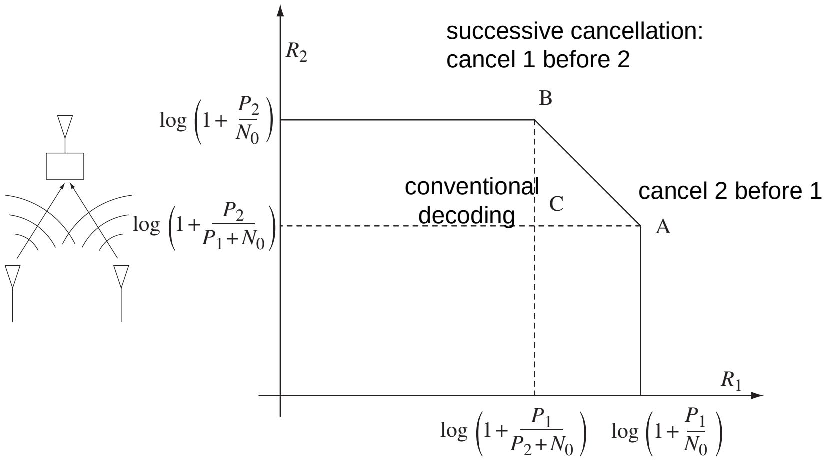
$$C \approx \frac{1.442}{1+f} \quad \text{bits/s/Hz}$$



Uplink and Downlink Capacity

- CDMA and OFDM are specific multiple access schemes.
- But information theory tells us what is the capacity of the uplink and downlink channels and the optimal multiple access schemes.

Uplink AWGN Capacity

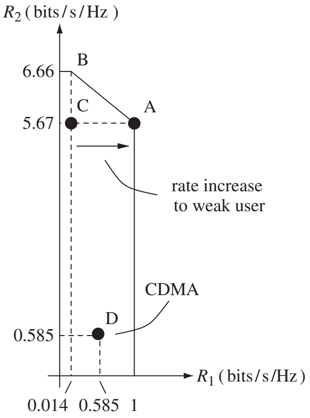


successive cancellation:
cancel 1 before 2

conventional
decoding

cancel 2 before 1

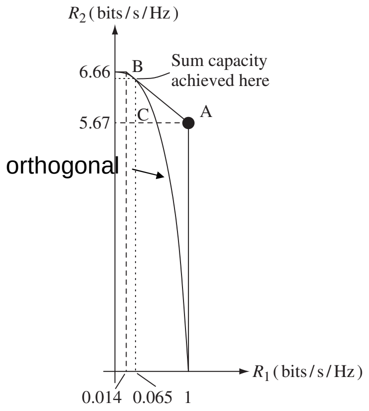
Conventional CDMA vs Capacity



20 dB power difference between 2 users

Successive cancellation allows the weak user to have a good rate without lowering the power of the strong user.

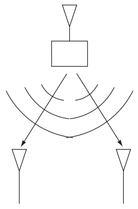
Orthogonal vs Capacity

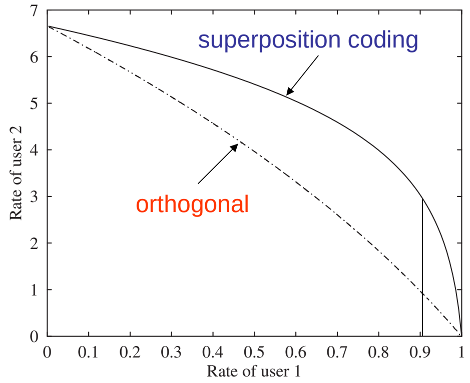


20 dB power difference between 2 users

Orthogonal achieves maximum throughput but may not be fair.

Downlink Capacity





20 dB gain difference between 2 users

Frequency-selective Channel

$$y[m] = \sum_{\ell} h_\ell x[m-\ell] + w[m]$$

h_l 's are time-invariant.

OFDM converts it into a *parallel channel*:

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \qquad n = 1, \ldots, N_c.$$

$$C_{N_c} = \sum_{n=0}^{N_c-1} \log\left(1 + \frac{P_n^* |\tilde{h}_n|^2}{N_0}\right),$$

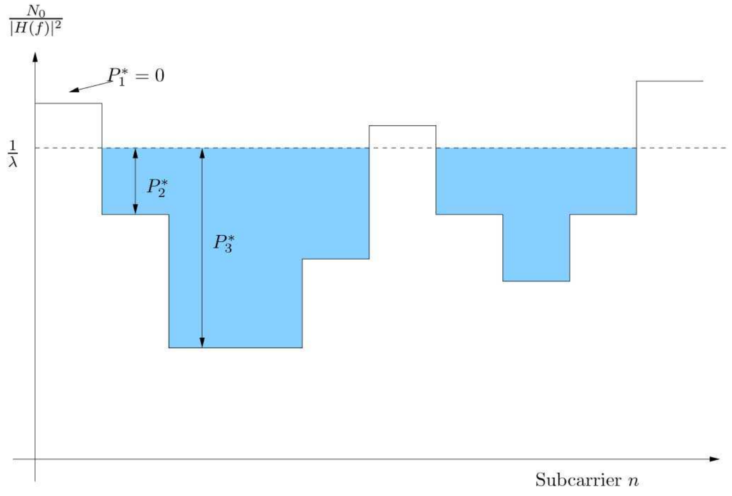
where P_n^* is the waterfilling allocation:

$$P_n^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_n|^2}\right)^+$$

with λ chosen to meet the power constraint.

Can be achieved with separate coding for each sub-carrier.

Waterfilling in Frequency Domain



Slow Fading Channel

$$y[m] = hx[m] + w[m]$$

h random.

There is no definite capacity.

Outage probability:

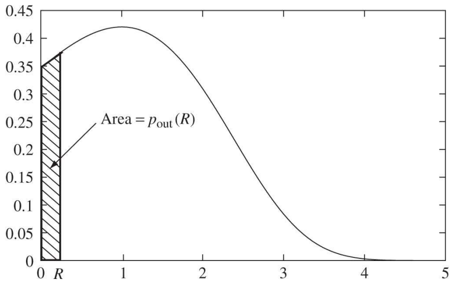
$$p_{\text{out}}(R) = \mathcal{P}\left\{\log(1+|h|^2\text{SNR}) < R\right\}$$

ϵ –outage capacity:

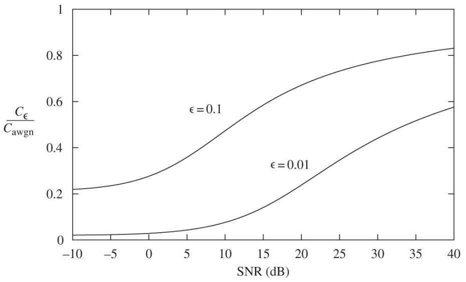
$$C_\epsilon = p_{\text{out}}^{-1}(\epsilon)$$

Outage for Rayleigh Channel

Pdf of log(1+|h|²SNR)



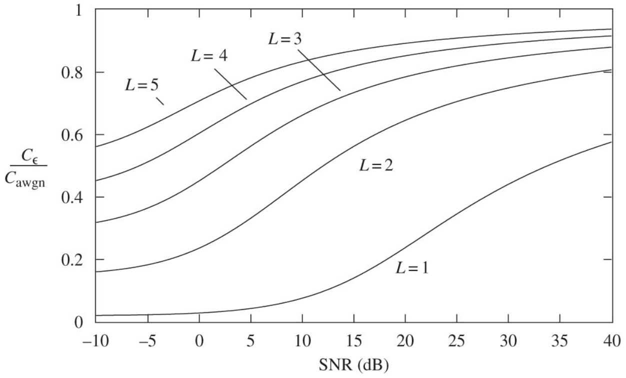
Outage cap. as fraction of AWGN cap.



$$p_{\text{out}}(R) \approx \frac{2^R - 1}{\text{SNR}}$$

Receive Diversity

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \text{SNR}\right) < R\right\}$$



Diversity plus power gain.

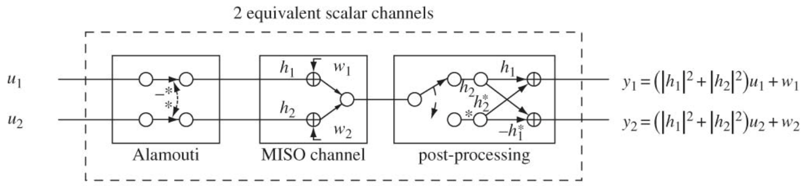
Transmit Diversity

Transmit beamforming:

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \text{SNR}\right) < R\right\}$$

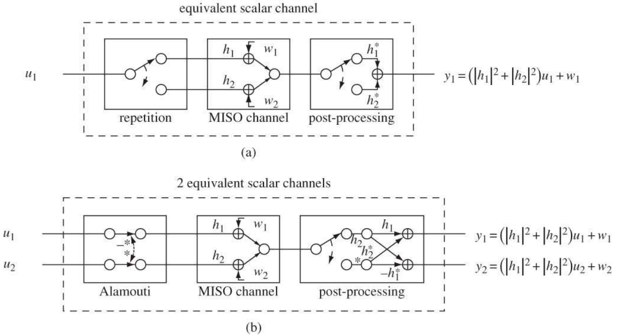
Alamouti (2 Tx):

$$p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \frac{\text{SNR}}{2}\right) < R\right\}$$



Diversity but no power gain.

Repetition vs Alamouti



Repetition: $p_{\text{out}}(R) = \mathcal{P}\left\{\frac{1}{2}\log\left(1 + \|\mathbf{h}\|^2 \text{SNR}\right) < R\right\}$

Alamouti: $p_{\text{out}}(R) = \mathcal{P}\left\{\log\left(1 + \|\mathbf{h}\|^2 \frac{\text{SNR}}{2}\right) < R\right\}$

Loss in degrees of freedom under repetition.

Time Diversity (I)

$$y_\ell = h_\ell x_\ell + w_\ell, \qquad \ell = 1, \ldots, L$$

Coding done over L coherence blocks, each of many symbols.

This is a parallel channel. If transmitter knows the channel, can do waterfilling.

Can achieve:

$$p_{\text{out}}(R) = \mathcal{P}\left\{\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1 + P_\ell^*|h_\ell|^2\right) < R\right\}$$

Time Diversity (II)

Without channel knowledge,

$$p_{\text{out}}(R) = \mathcal{P}\left\{\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1+|h_\ell|^2\text{SNR}\right) < R\right\}$$

Rate allocation cannot be done.

Coding across sub-channels becomes now necessary.

Fast Fading Channel

Channel with L-fold time diversity:

$$p_{\text{out}}(R) = \mathcal{P}\left\{\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1+|h_\ell|^2\text{SNR}\right) < R\right\}$$

As $L \to \infty$,

$$\frac{1}{L}\sum_{\ell=1}^{L}\log\left(1+|h_\ell|^2\text{SNR}\right) \to \mathcal{E}[\log(1+|h|^2\text{SNR})]$$

Fast fading channel has a definite capacity:

$$C = \mathcal{E}[\log(1+|h|^2\text{SNR})]$$

Tolerable delay >> coherence time.

Capacity with Full CSI

Suppose now transmitter has full channel knowledge.

What is the capacity of the channel?

Fading Channel with Full CSI

This is a parallel channel, with a sub-channel for each fading state.

$$C = \mathcal{E}\left[\log\left(1 + \frac{P^*(h)|h|^2}{N_0}\right)\right]$$

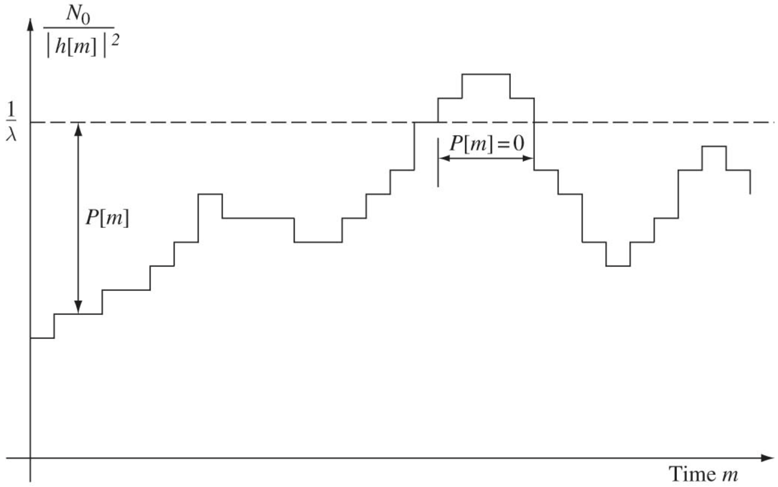
where

$$P^*(h) = \left(\frac{1}{\lambda} - \frac{N_0}{|h|^2}\right)^+.$$

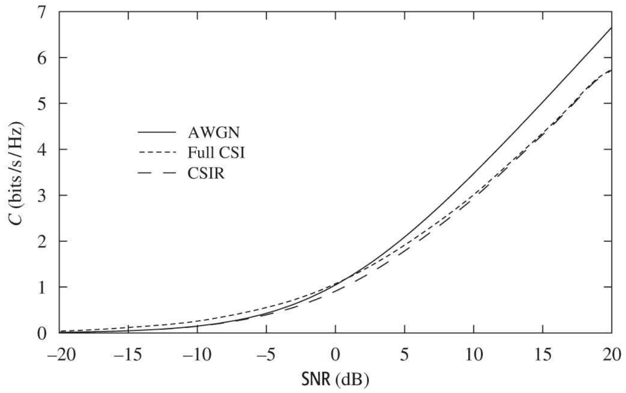
is the waterfilling power allocation as a function of the fading state, and λ is chosen to satisfy the average power constraint.

Can be achieved with separate coding for each fading state.

Transmit More when Channel is Good

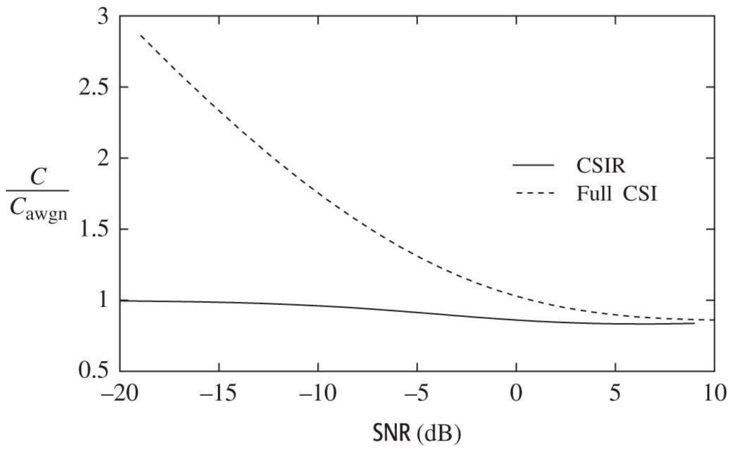


Performance



At high SNR, waterfilling does not provide any gain.
But transmitter knowledge allows rate adaptation and simplifies coding.

Performance: Low SNR

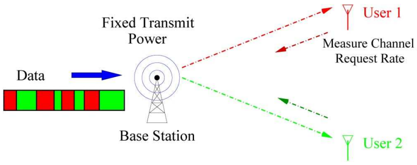


Waterfilling povides a significant power gain at low SNR.

Waterfilling vs Channel Inversion

- Waterfilling and rate adaptation maximize long-term throughput but incur significant delay.
- Channel inversion (“perfect” power control in CDMA jargon) is power-inefficient but maintains the same data rate at all channel states.
- Channel inversion achieves a delay-limited capacity.

Example of Rate Adaptation: 1xEV-DO Downlink



Multiple access is TDMA via scheduling. (More on this tomorrow.)

Each user is rate-controlled rather than power-controlled. (But no waterfilling.)

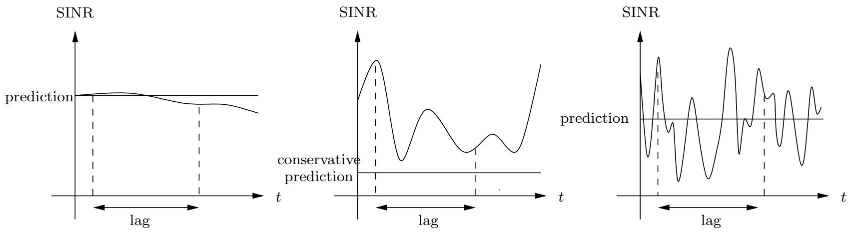
Rate Control

Mobile measures the channel based on the pilot and predicts the SINR to request a rate.

Requested rate (kbits/s)	SINR threshold (dB)	Modulation	Number of slots
38.4	−11.5	QPSK	16
76.8	−9.2	QPSK	8
153.6	−6.5	QPSK	4
307.2	−3.5	QPSK	2 or 4
614.4	−0.5	QPSK	1 or 2
921.6	2.2	8-PSK	2
1228.8	3.9	QPSK or 16-QAM	1 or 2
1843.2	8.0	8-PSK	1
2457.6	10.3	16-QAM	1

SINR Prediction Uncertainty

$f_c = 1.9$ GHz



(a) 3 km/hr

(b) 30 km/hr

(c) 120 km/hr

accurate prediction of instantaneous SINR.

conservative prediction of SINR.

accurate prediction of average SINR for a fast fading channel

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Incremental ARQ

- A conservative prediction leads to a lower requested rate.
- At such rates, data is repeated over multiple slots.
- If channel is better than predicted, the number of repeated slots may be an overkill.
- This inefficiency can be reduced by an incremental ARQ protocol.
- The receiver can stop transmission when it has enough information to decode.
- Incremental ARQ also reduces the power control accuracy requirement in the reverse link in Rev A.

Summary

- A slow fading channel is a source of unreliability: very poor outage capacity. Diversity is needed.
- A fast fading channel with only receiver CSI has a capacity close to that of the AWGN channel. Delay is long compared to channel coherence time.
- A fast fading channel with full CSI can have a capacity *greater* than that of the AWGN channel: fading now provides more *opportunities* for performance boost.
- The idea of *opportunistic communication* is even more powerful in multiuser situations, as we will see.