ECE 312 HW #4 solution

1.

The Fourier transform of $x(t) = e^{-2t}u(t)$ is

$$X(j\omega) = \frac{1}{2 + j\omega}.$$

The Fourier transform of the output signal $y(t)$ is

$$Y(j\omega) = \frac{1}{1 + j\omega} - \frac{1}{3 + j\omega}.$$

Since system S is LTI, we can use the convolution property of the Fourier transform to get its frequency response:

$$\begin{aligned} H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ &= \frac{2 + j\omega}{1 + j\omega} - \frac{2 + j\omega}{3 + j\omega} \\ &= \frac{6 + 5j\omega - \omega^2 - 2 - 3j\omega + \omega^2}{(1 + j\omega)(3 + j\omega)} \\ &= \frac{4 + 2j\omega}{(1 + j\omega)(3 + j\omega)}. \end{aligned}$$

Now, in order to find the impulse reponse of system S, we need to find the partial fraction expansion of $H(j\omega)$. We set up the following equation:

$$\frac{4 + 2j\omega}{(1 + j\omega)(3 + j\omega)} = \frac{A}{1 + j\omega} + \frac{B}{3 + j\omega},$$

and then we solve for A and B. To do that, we multiply both sides by $(1 + j\omega)(3 + j\omega)$ to get:

$$4 + 2j\omega = 3A + B + (A + B)j\omega.$$

Comparing both sides, we get that $A + B = 2$ and $3A + B = 4$. Thus, $A = B = 1$. So now we have that

$$H(j\omega) = \frac{1}{1 + j\omega} + \frac{1}{3 + j\omega}.$$

Taking the inverse Fourier transform of the above equation, we get:

$$h(t) = e^{-t}u(t) + e^{-3t}u(t).$$

2.

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\begin{aligned}
x_1(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi e^{-2|\omega|} \cdot e^{j\omega t} d\omega \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{2\omega} e^{j\omega t} d\omega + \frac{1}{2}\int_{0}^{\infty} e^{-2\omega} e^{j\omega t} d\omega \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{(2+jt)\omega} d\omega + \frac{1}{2}\int_{0}^{\infty} e^{(jt-2)\omega} d\omega \\
&= \frac{1}{2(2+jt)} - \frac{1}{2(jt-2)} \\
&= \frac{2-jt}{2(4+t^2)} + \frac{2+jt}{2(4+t^2)} \\
&= \frac{2}{4+t^2}.
\end{aligned}
$$

3.

$$X(\omega) = \frac{e^{-j\omega}}{a + j\omega}$$