**Table 4.5** *Basic Root Locus Principles for Negative Parameters*

1. The branches of the locus are continuous curves that start at each of the $n$ poles of $GH$, for $K = 0$. As $K$ approaches $-\infty$, the locus branches approach the $m$ zeros of $GH$. Locus branches for excess poles extend infinitely far from the origin; for excess zeros, locus segments extend from infinity.
2. The locus includes all points along the real axis to the left of an even number of poles plus zeros of $GH$.
3. As $K$ approaches $-\infty$, the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{i360°}{n - m}$$

for $i = 0, \pm 1, \pm 2, \ldots,$ until all $n - m$ or $m - n$ angles are obtained, where $n$ is the number of poles and $m$ is the number of zeros of $GH$.

4. The starting point of the asymptotes, the centroid of the pole-zero plot, is on the real axis at

$$\sigma = \frac{\Sigma \text{ pole values of } GH - \Sigma \text{ zero values of } GH}{n - m}$$

5. Loci leave the real axis at a gain $K$ that is the maximum $K$ in that region of the real axis. Loci enter the real axis at the minimum value of $K$ in that region of the real axis. These points are termed *breakaway points* and *entry points*, respectively. A pair of locus segments leave or enter the real axis at angles of $\pm 90°$.
6. The angle of departure $\phi$ of a locus branch from a complex pole is given by

$$\phi = -\Sigma \text{ other } GH \text{ pole angles} + \Sigma\, GH \text{ zero angles} + 0°$$

ZERO

The angle of approach $\phi'$ of a locus branch to a complex pole is given by

$$\phi' = \Sigma\, GH \text{ pole angles} - \Sigma \text{ other } GH \text{ zero angles} - 0°$$

where each $GH$ pole angle and $GH$ zero angle is calculated to the complex pole for $\phi$ and to the complex zero for $\phi'$.

If the complex pole or zero is of order $m$, the $m$ angles of arrival and approach are given by

$$\phi = [-\Sigma \text{ other } GH \text{ pole angles} + \Sigma\, GH \text{ zero angles} + i360°]/m$$

$$\phi' = [\Sigma\, GH \text{ pole angles} - \Sigma \text{ other } GH \text{ zero angles} - i360°]/m$$

for $i = 0, 1, 2, \ldots, (m - 1)$.

The root locus method may also be applied to nonstandard systems by first converting them to the standard form

$$T(s) = \frac{\text{num. poly.}}{1 + K\frac{a(s)}{b(s)}}$$

In addition, root locus plots can be obtained for negative values of the parameter K.

---

Example:

Determine the root locus of the following function as K is varied from minus to plus infinity.

$$T(s) = \frac{6s^2 + Ks + 2}{s^3 + 2s^2 + (5+2K)s + 2K}$$

---

Converting T(s) to the standard form,

$$T(s) = \frac{6s^2 + Ks + 2}{s^3 + 2s^2 + 5s + K(2s+2)}$$

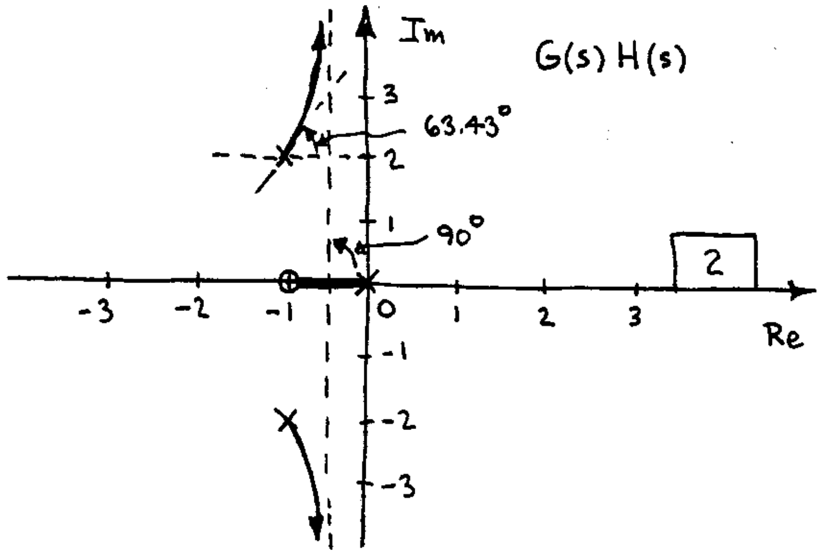
$$= \frac{\dfrac{6s^2 + Ks + 2}{s^3 + 2s^2 + 5s}}{1 + K\dfrac{2s+2}{s^3 + 2s^2 + 5s}}$$

The equivalent G(s) H(s) for the root locus is

$$G(s)\,H(s) = \frac{2(s+1)}{s(s^2+2s+5)}$$

---

For $0 \le K < \infty$,




Applying basic root locus principles,

$$\theta = \frac{180^\circ \pm i\,360^\circ}{2} = \underline{\pm 90^\circ}$$

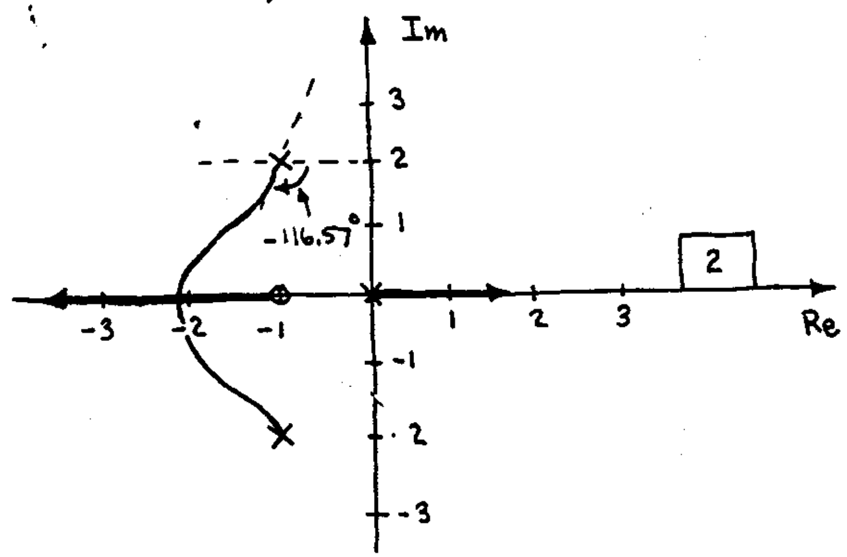
$$\sigma = \frac{-1+j2-1-j2-(-1)}{2} = \underline{-\frac{1}{2}}$$

$$116.57^\circ + 90^\circ + \phi - 90^\circ = 180^\circ$$

$$\phi = \underline{63.43^\circ}$$

---

For $-\infty < K \leq 0$,




And,

$$\theta = \frac{0° \pm i\,360°}{2} = \underline{0°,\ 180°}$$

$$\sigma = \underline{-\frac{1}{2}}$$

$$116.57° + 90° + \phi - 90° = 0°$$

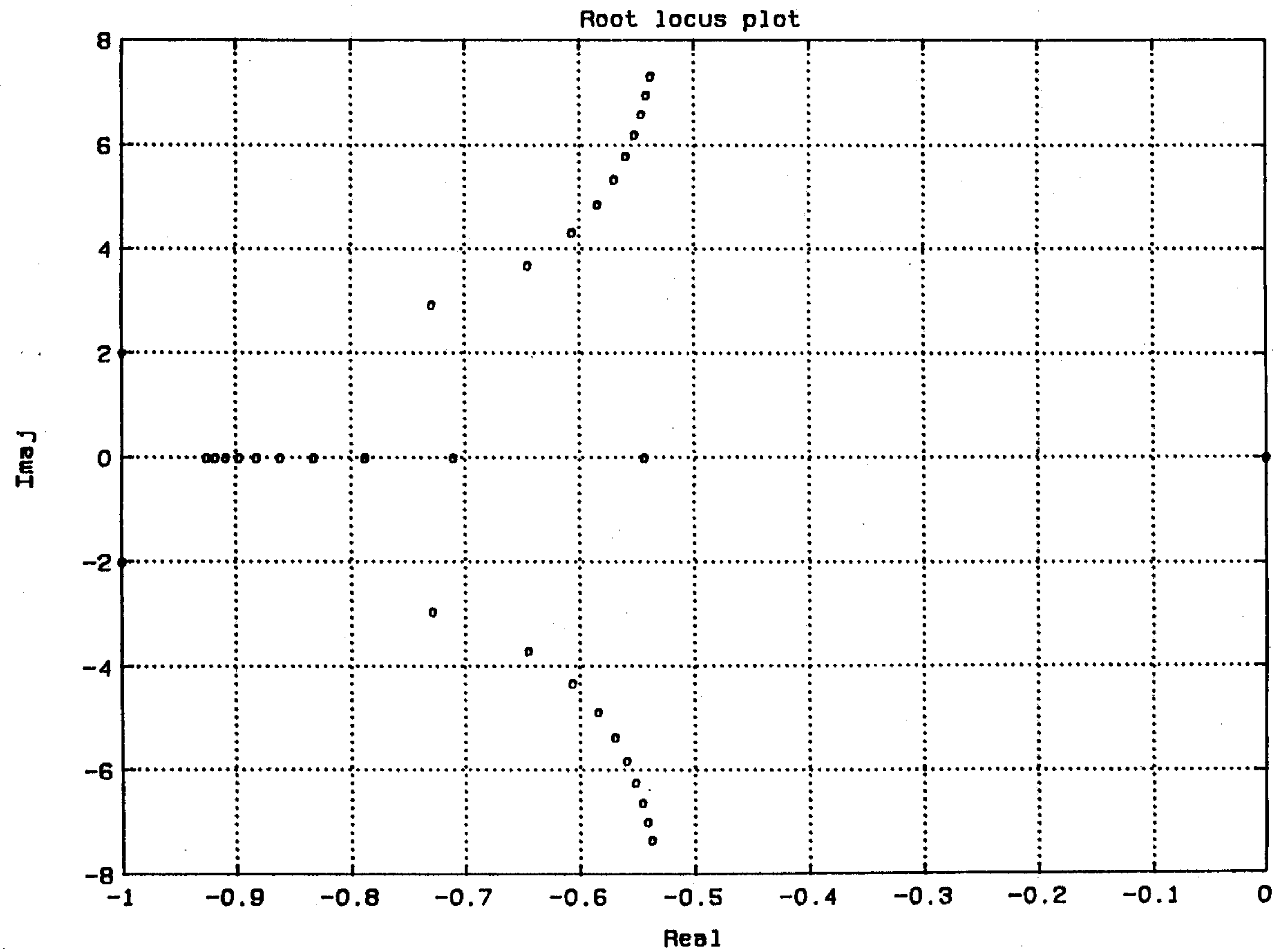
$$\phi = \underline{-116.57°}$$

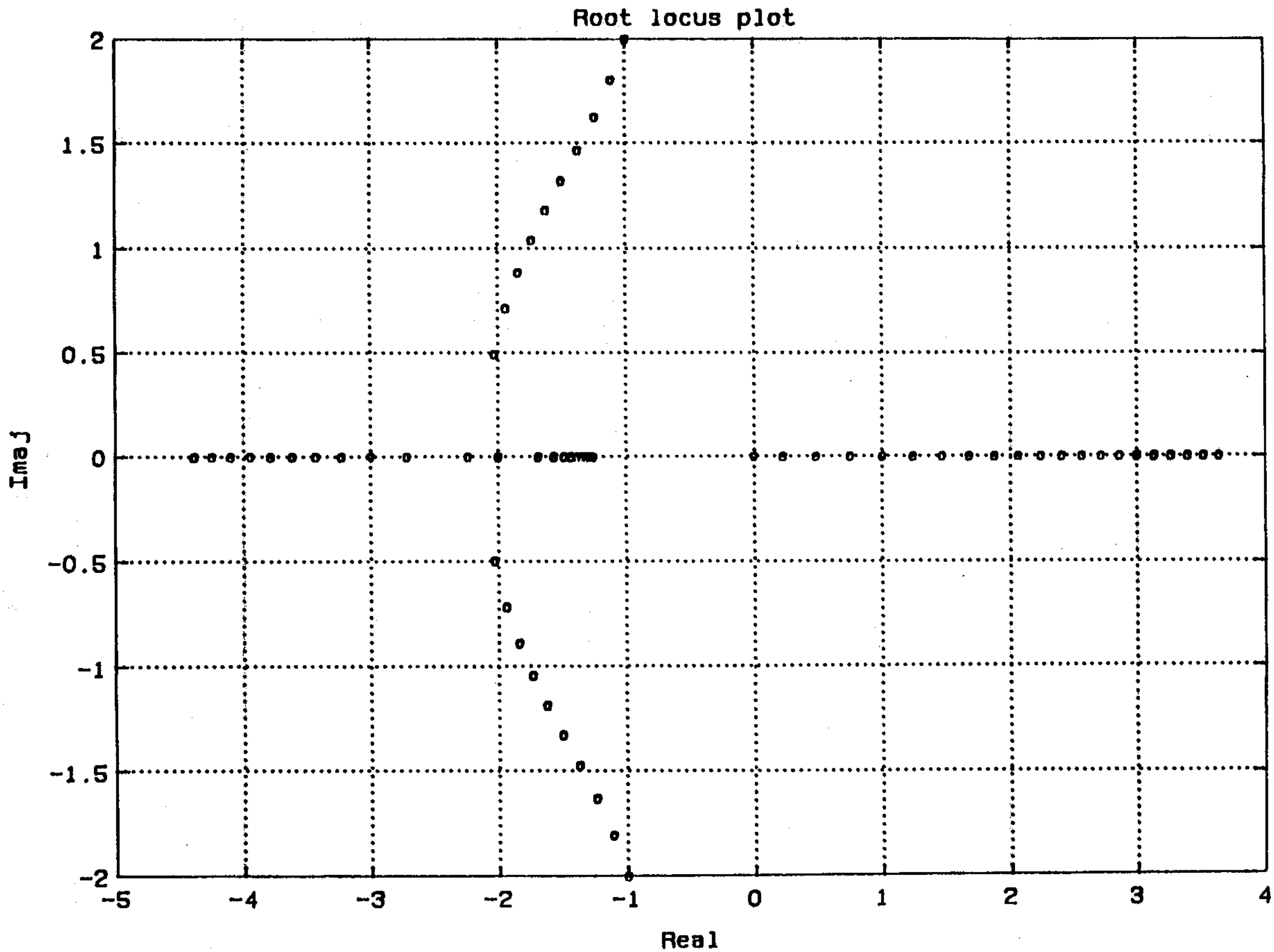
The real axis entry point is determined by evaluating

$$K = \left| \frac{s(s^2+2s+5)}{2(s+1)} \right|$$

to yield

| Value of s | Value of K |
|---|---|
| -2.3 | 5.03 |
| -2.2 | 4.99 |
| -2.1 | 4.97 |
| -2.0 | 5.00 |


Root locus plot
Imaj
Real
8
6
4
2
0
-2
-4
-6
-8
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0


Root locus plot
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
Imag
-5
-4
-3
-2
-1
0
1
2
3
4
Real

# Root Locus Design

When designing control systems using root locus methods, it is important to understand the relationship between the root locus and the resulting time response.

---

Consider the following second-order closed-loop transfer function expressed in standard form including the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If the damping ratio range is $0 < \zeta < 1$, the complex conjugate poles of $T(s)$ are

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \sigma + j\omega)(s + \sigma - j\omega)$$

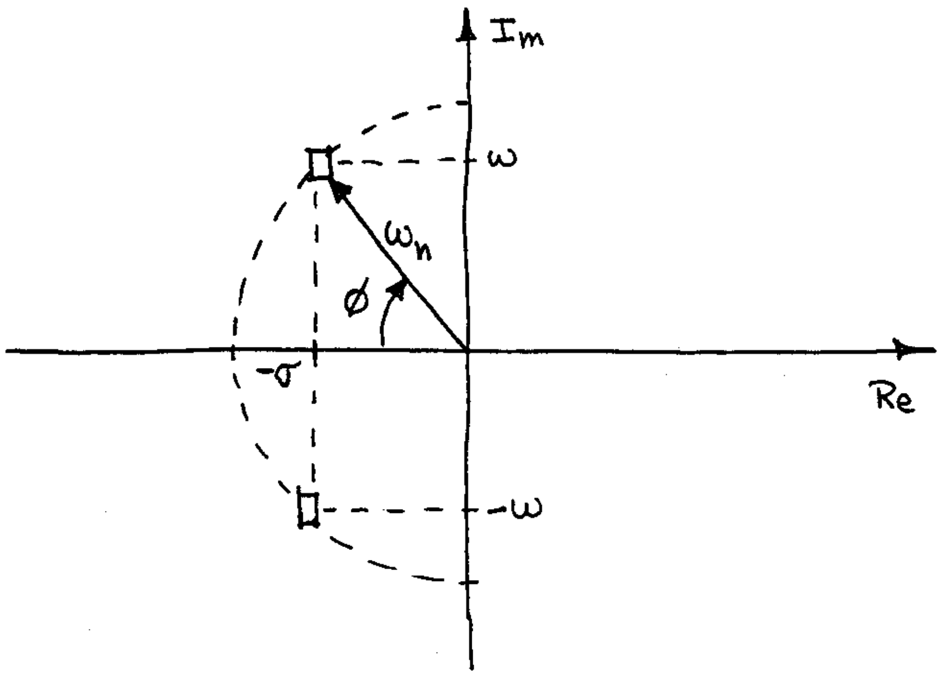
$$= s^2 + 2\sigma s + \sigma^2 + \omega^2$$

Comparing terms,

$$2\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{\sigma}{\omega_n}$$

If the closed-loop poles of T(s) are graphed as boxes,

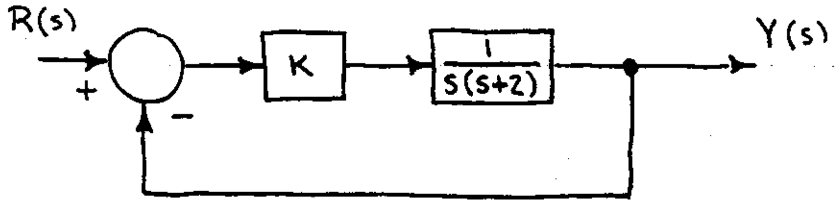



and

$$\cos\phi = \zeta = \frac{\sigma}{\omega_n}$$

Example :

Determine the rise time and settling time for the following control system when $K = 4$.

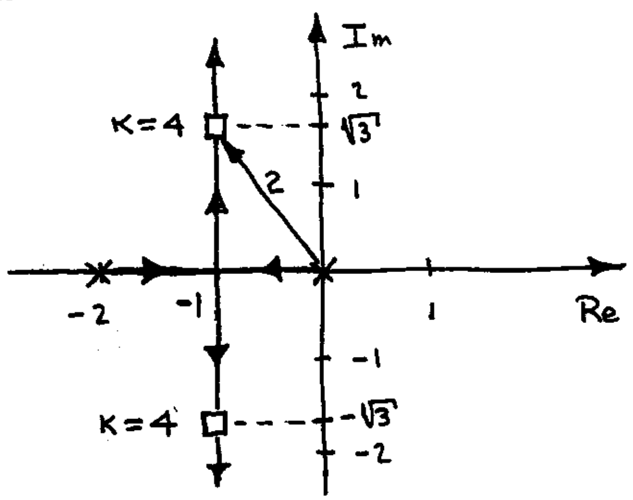



The transfer function is

$$T(s) = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s(s+2)+K}$$

$$= \frac{4}{s^2+2s+4} = \frac{4}{(s+1+j\sqrt{3})(s+1-j\sqrt{3})}$$

The root locus is



From the root locus plot, $\omega_n = 2$ and $\zeta = \frac{1}{2}$. Therefore, from the tables,

$$\omega_n T_r = 1.7$$

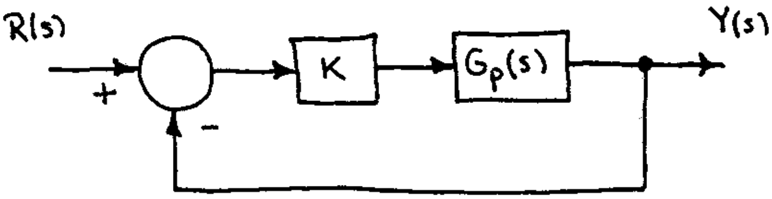
$$T_r = \frac{1.7}{\omega_n} = \frac{1.7}{2} = \underline{\underline{0.85\ s}} \leftarrow$$

$$\omega_n T_s = 5.2$$

$$T_s = \frac{5.2}{\omega_n} = \frac{5.2}{2} = \underline{\underline{2.6\ s}} \leftarrow$$

# Compensation

The simplest tracking control system employs a unity feedback loop around a plant $G_p(s)$ cascaded with an error signal amplifier with gain $K$.




The amplifier gain is typically adjusted to provide acceptable performance of the closed-loop system's relative stability and steady state error performance. If adequate performance cannot be obtained with output feedback alone, additional transmittances termed <u>compensators</u> may be added to the system.

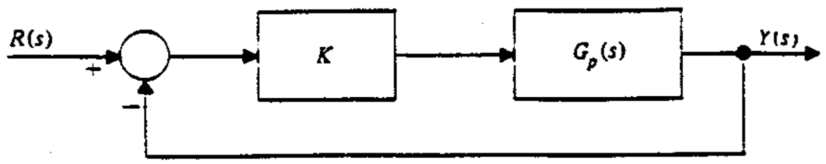



**Figure 5.6** Unity feedback control system.

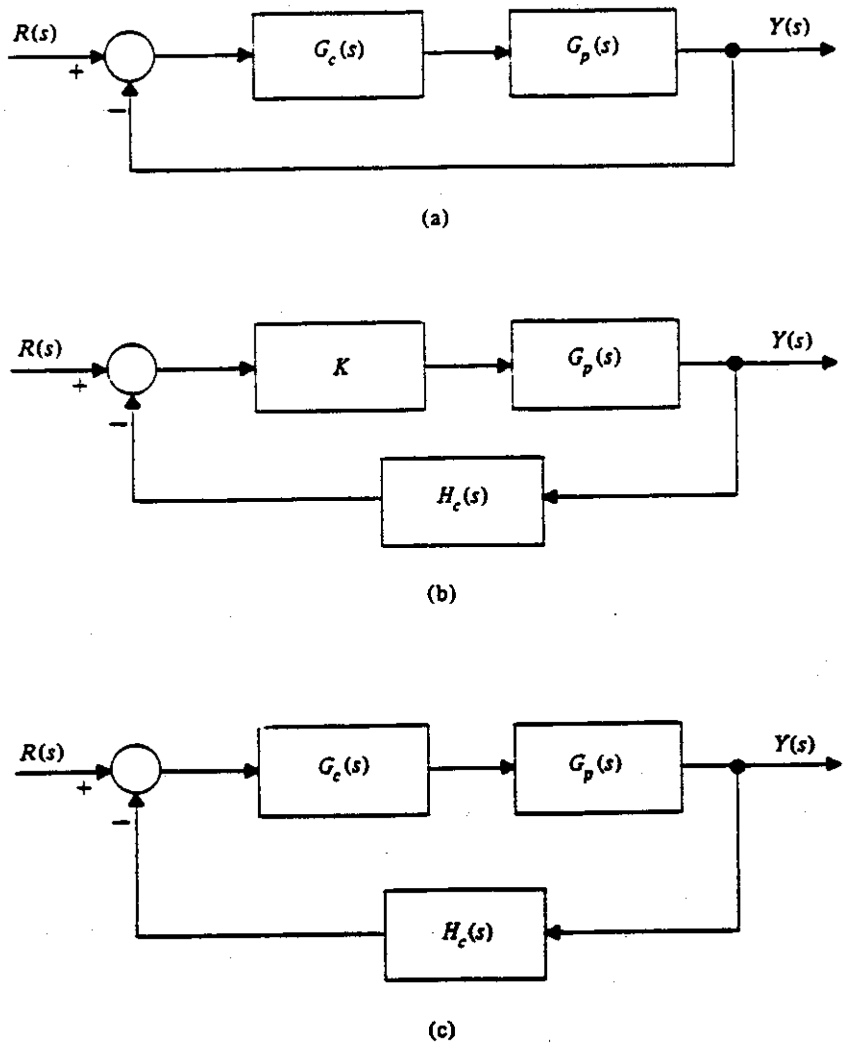



**Figure 5.7** Compensator configurations. **(a)** Cascade compensated system. **(b)** Feedback compensated system. **(c)** System with feedback and cascade compensation.

*Table 5.1 Common Types of Compensators*

| Compensator | Transmittance | Typical Effect on Steady State Errors | Typical Effect on Relative Stability |
|---|---|---|---|
| Cascade PI | $G_c(s) = \frac{K(s + a)}{s}$ | Greatly improved | Reduced |
| Cascade lag | $G_c(s) = \frac{K(s + a)}{s + b}$ <br> $b < a$ | Improved | Reduced |
| Cascade lead | $G_c(s) = \frac{K(s + a)}{(s + b)}$ <br> $a < b$ | Somewhat improved or somewhat worse | Increased |
| Cascade lag-lead | $G_c(s) = K\left(\frac{s + a}{s + b}\right)_{\text{lag}} \times \left(\frac{s + a}{s + b}\right)_{\text{lead}}$ | Improved | Increased |
| Rate feedback (PD) | $H_c(s) = 1 + As$ | Somewhat improved or somewhat worse | Increased |
| Proportional integral derivative (PID) | $G_c(s) = \frac{K(s + a)}{s} + KAs$ | Greatly improved | Increased |

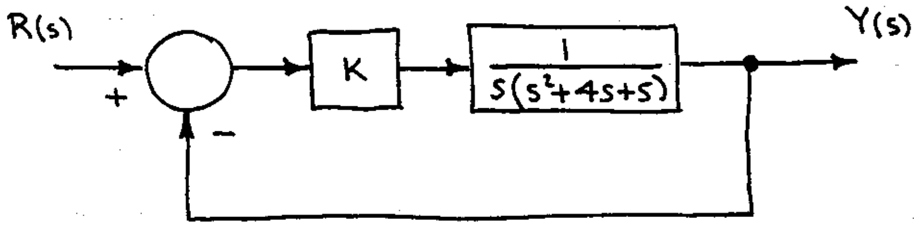
SMALL S

*Table 5.2 Common Types of Compensators*

| Compensator | Change in $n - m$ | Change in Centroid if $n - m$ is Constant | Change in Type Number | $\frac{K_i(\text{comp})}{K_i(\text{uncomp})}$ |
|---|---|---|---|---|
| Cascade PI | 0 | $\frac{a}{n - m}$ | +1 | $\infty$ |
| Cascade lag ($a > b$) | 0 | $\frac{a - b}{n - m}$ | 0 | $\frac{K}{K_u}\left(\frac{a}{b}\right)$ |
| Cascade lead ($b > a$) | 0 | $\frac{-(b - a)}{n - m}$ | 0 | $\frac{K}{K_u}\left(\frac{a}{b}\right)$ |
| Cascade lag-lead | 0 | $\frac{(a - b)_{\text{lag}} - (b - a)_{\text{lead}}}{n - m}$ | 0 | $\frac{K}{K_u}\left(\frac{a}{b}\right)_{\text{lag}}\left(\frac{a}{b}\right)_{\text{lead}}$ |
| Rate feedback (PD) | −1 | | 0 | $\frac{1}{K_u/K + AK_1(\text{uncomp})}$ |
| Proportional integral derivative | −1 | | +1 | $\infty$ |

Example :

Consider the following uncompensated system with feedback.



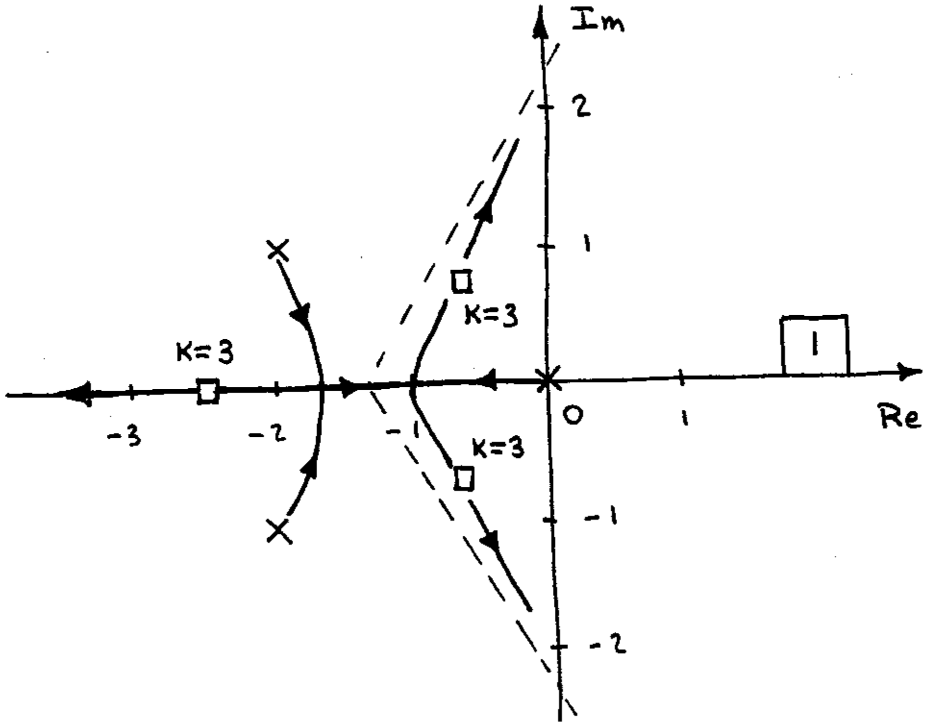

The plant transmittance is

$$G_p(s) = \frac{1}{s(s^2+4s+5)} = \frac{1}{s(s+2+j)(s+2-j)}$$

Root locus information is :

1. Poles at $s = 0, -2 \pm j$
2. Real axis locus segment $s \leq 0$
3. Asymptotic angles $\Theta = \pm 60^\circ, 180^\circ$
4. Asymptotic centroid $\sigma = -1.33$
5. Breakaway point $s = -1.0$, entry point $s = -1.7$
6. Angle of departure (top pole) $\phi = -63.43^\circ$

The root locus is




If $K = 3$, the closed-loop poles as indicated on the root locus are

$$1 + KG_p(s) = 1 + \frac{3}{s(s^2+4s+5)} = 0$$

$\therefore$

$$s^3 + 4s^2 + 5s + 3 = 0$$

$$(s+2.47)(s+0.767 \pm j0.793) = 0$$

$$s = -2.47, \ -0.767 \pm j0.793$$

In addition, if $\omega = 0.793$ and $\sigma = 0.767$,

$$\omega_n = 1.10$$

$$\delta = 0.70$$

The ramp error coefficient of this type 1 system is

$$K_1 = \lim_{s \to 0} s K G_p(s) = \frac{K}{5} = \frac{3}{5} = 0.6$$

For a unit ramp input, the steady state error is

$$\lim_{t \to \infty} e_{ramp}(t) = \frac{1}{K_1} = \underline{\underline{1.67}} \leftarrow$$

As indicated above, the relative stability is $\underline{\underline{0.767}}$. ←

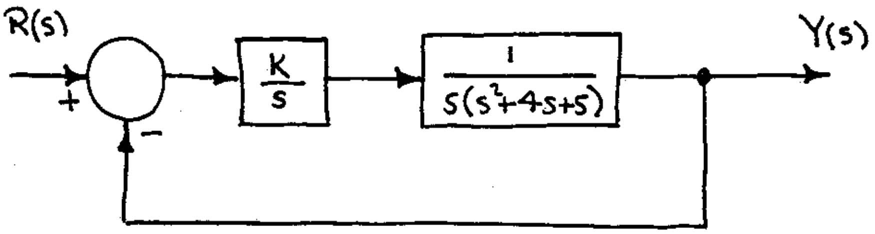
As the gain K is varied in the design, it is important to realize that steady state error and relative stability are direct trade-offs. A reduction in steady state error will be accompanied by a reduction in relative stability. Similarly, an increase in relative stability will result in an increase in steady state error.

The normal design strategy is to place compensator poles near the origin to improve steady state performance and to place compensator zeroes so that the system root locus gives sufficient relative stability for a suitable value of the compensator multiplying constant K.

---

Example :

Consider the following cascade integral compensated system.

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The cascade integral compensator adds a single pole at $s = 0$ to the system's open-loop transmittance, thus raising the system type number.

The ramp error coefficient of this type 2 system is

$$K_2 = \lim_{s \to 0} s\, G_c(s)\, G_p(s)$$

$$= \lim_{s \to 0} \frac{K}{s(s^2+4s+5)} = \infty$$

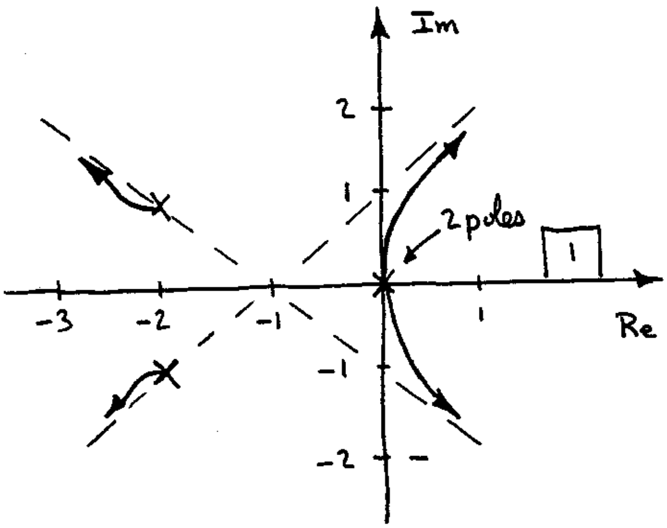
indicating the steady state error is zero.

---

Root locus information is :

1. Poles at $s = 0, 0, -2 \pm j$
2. No real axis locus segments
3. Asymptotic angles $\theta = \pm 45°, \pm 135°$
4. Asymptotic centroid $\sigma = -1$
5. Breakaway point $s = 0$
6. Angle of departure (top pole) $\phi = -216.86°$

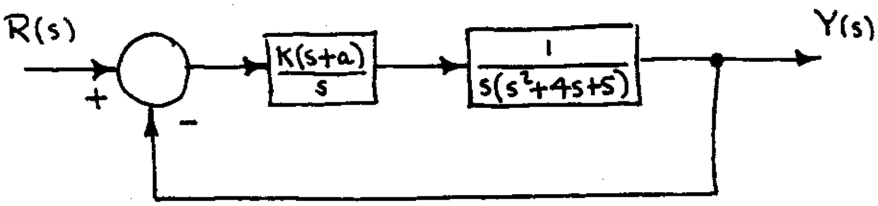
The root locus is




This compensated system is unstable for all positive K.

Example:

Consider the following cascade integral plus proportional compensated system.




The cascade integral plus proportional compensator adds a single pole at $s = 0$ and a zero at $s = -a$. System type increases by one but there is no change in $n-m$. therefore the compensated system has the same asymptotic angles as the un-compensated system. However, for positive $a$, the centroid $\sigma$ is moved to the right.

Both parameters $K$ and $a$ may be adjusted in the design, and their selection is usually alternated in on iterative manner until a satisfactory design results. If $a = 0$ is used as a starting point for the iterative procedure, the compensator transmittance reduces to $G_c(s) = K$, which is that for the uncompensated feedback system.

The compensated system transfer function is

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

$$= \frac{\dfrac{K(s+a)}{s}\left[\dfrac{1}{s(s^2+4s+5)}\right]}{1 + \dfrac{K(s+a)}{s}\left[\dfrac{1}{s(s^2+4s+5)}\right]}$$

$$= \frac{\dfrac{K(s+a)}{s^2(s^2+4s+5)}}{1 + K\dfrac{s+a}{s^2(s^2+4s+5)}}$$ $\left\{\text{Root locus for fixed } a \text{ and variable } K\right.$

$$= \frac{K(s+a)}{s^2(s^2+4s+5)+Ks+Ka}$$

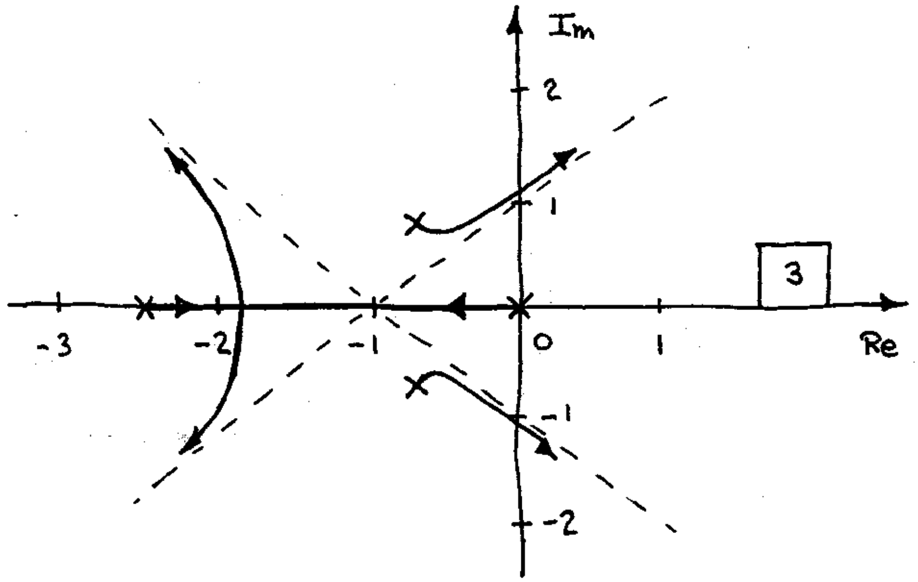
$$= \frac{\dfrac{K(s+a)}{s(s^3+4s^2+5s+K)}}{1 + a\dfrac{K}{s(s^3+4s^2+5s+K)}}$$ $\left\{\text{Root locus for fixed } K \text{ and variable } a\right.$

With K equal to the uncompensated value of three and a adjustable, two of the open-loop poles are the dominant closed-loop poles of the uncompensated system.

Root locus information is :

1. Poles at $0, -2.47, -0.767 \pm j0.793$
2. Real axis locus segment $-2.47 \leq s \leq 0$
3. Asymptotic angles $\theta = \pm 45°, \pm 135°$
4. Asymptotic centroid $\sigma = -1.00$
5. Breakaway point $s = -1.88$
6. Angle of departure (top pole) $\phi = -69.02°$

The root locus is




As a increases, the root locus migrates to the right from the complex poles, indicating a relatively small value of a is appropriate. The root locus moves left from the origin for all positive a.

If $a = 0.1$, the closed loop poles are

$$T(s) = \frac{3(s+0.1)}{s^4 + 4s^3 + 5s^2 + 3s + 0.3}$$

$\therefore$

$$s^4 + 4s^3 + 5s^2 + 3s + 0.3 = 0$$

$$(s + 0.123)(s + 2.43)(s + 0.724 \pm j0.694) = 0$$

$$s = -0.123, -2.43, -0.724 \pm j0.694$$

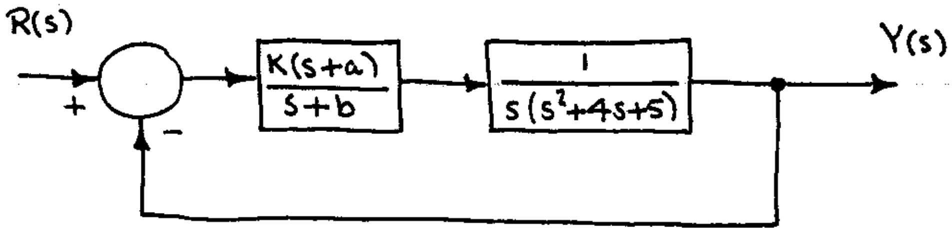
It is important to realize that the closed-loop pole at $s = -0.123$ is nearly cancelled by the closed-loop zero at $s = -0.1$, and therefore the natural response component due to this pole is very small. Thus the dominant roots are $\omega = 0.694$ and $\sigma = 0.724$, producing

$$\omega_n = 1.00$$
$$\zeta = 0.72$$

The ramp error coefficient of this type 2 system is $K_2 = \infty$ and the steady state error is zero.

Example :

Consider a cascade lag compensator




where K, a, and b are real positive constants and $a > b$.

For nonzero b, the system type number is not increased. However, the steady state error performance can be improved over that of the uncompensated feedback system.

Beginning with the error coefficient for an uncompensated system

$$K_i = K \lim_{s \to 0} s^i G_p(s)$$

the error coefficient for a lag compensated system is

$$K_i = \lim_{s \to 0} s^i G_c(s)\, G_p(s)$$

$$= K\left(\frac{a}{b}\right) \lim_{s \to 0} s^i G_p(s)$$

Therefore,

$$\frac{K_i \text{ (comp.)}}{K_i \text{ (uncomp.)}} = \frac{K \text{ (comp.)}}{K \text{ (uncomp.)}} \left(\frac{a}{b}\right)$$

If $K \text{ (comp.)} = K \text{ (uncomp.)}$,

$$K_i \text{ (comp.)} = K_i \text{ (uncomp.)} \left(\frac{a}{b}\right)$$

The centroid for a cascade lag compensated system moves to the right by

$$\frac{a-b}{n-m}$$

indicating a reduction in relative stability as steady state error performance is improved. If a is small ( $a = 0.1$ ) and b is selected to be one-tenth of a ( $b = 0.01$ ), then

$$\frac{a-b}{n-m} = \frac{0.1 - 0.01}{n-m} = \frac{0.09}{n-m}$$

and the centroid moves only a very small amount to the right.

The ramp error coefficient for the compensated system is

$$K_1 \text{ (comp.)} = K_1 \text{ (uncomp.)} \left(\frac{a}{b}\right)$$

$$= 0.6\,(10) = 6$$

The closed-loop poles of $T(s)$ are

$$T(s) = \frac{3(s+0.1)}{s^4+4.01s^3+5.04s^2+3.05s+0.3}$$

$\therefore$

$$s^4 + 4.01s^3 + 5.04s^2 + 3.05s + 0.3 = 0$$

$$(s+0.120)(s+2.43)(s+0.729 \pm j0.705) = 0$$

$$s = -0.120, -2.43, -0.729 \pm j0.705$$

Again the closed-loop pole at $s = -0.120$ is nearly cancelled by the closed-loop zero at $s = -0.1$, and the transient response is dominated by the closed-loop poles $s = -0.729 \pm j0.705$.

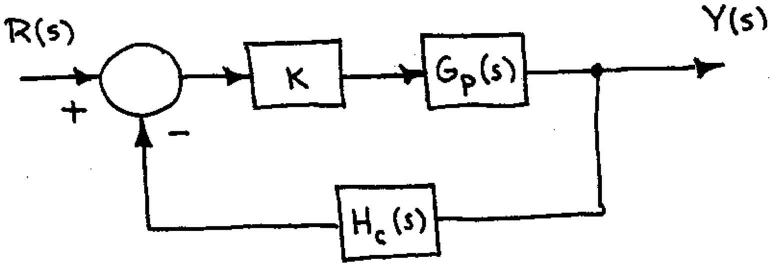
For these dominant roots, $\omega = 0.705$ and $\sigma = 0.729$, and

$$\omega_n = 1.01$$

$$\zeta = 0.72$$

Example :

Consider a feedback compensated system




where $H_c(s) = 1 + As$, producing feedback rate compensation.

The compensated system transfer function is

$$T(s) = \frac{KG_p(s)}{1 + KG_p(s)\,H_c(s)}$$

and the equivalent forward path transmittance for unity feedback is

$$G_E(s) = \frac{KG_p(s)}{1 + KG_p(s)\left[H_c(s) - 1\right]}$$

$$= \frac{KG_p(s)}{1 + KG_p(s)\left[1 + As - 1\right]}$$

$$= \frac{KG_p(s)}{1 + AKs\,G_p(s)}$$

The ith error coefficient is then

$$K_i = \lim_{s \to 0} s^i G_E(s)$$

Now assuming that K in the above block diagram represents K(comp.), then

$$K_i(\text{comp.}) = \lim_{s \to 0} \frac{K(\text{comp.})\, s^i G_p(s)}{1 + A K(\text{comp.})\, s\, G_p(s)}$$

$$= \lim_{s \to 0} \frac{K(\text{comp.})\, s^i K(\text{uncomp.})\, G_p(s)}{K(\text{uncomp.}) + A K(\text{comp.})\, s\, K(\text{uncomp.})\, G_p(s)}$$

$$= \frac{K(\text{comp.})\, K_i(\text{uncomp.})}{K(\text{uncomp.}) + A K(\text{comp.})\, K_1(\text{uncomp.})}$$

Therefore,

$$\frac{K_i(\text{comp.})}{K_i(\text{uncomp.})} = \frac{K(\text{comp.})}{K(\text{uncomp.}) + A K(\text{comp.})\, K_1(\text{uncomp.})}$$

If the uncompensated open-loop forward transmittance is

$$K(\text{uncomp.})\,G_p(s) = \frac{3}{s(s^2+4s+5)}$$

then

$$K_1(\text{uncomp.}) = \lim_{s \to 0} s\,K(\text{uncomp.})\,G_p(s)$$

$$= 0.6$$

and

$$\frac{K_1(\text{comp.})}{K_1(\text{uncomp.})} = \frac{K(\text{comp.})}{3 + A\,K(\text{comp.})\,(0.6)}$$

If $K(\text{comp.}) = 6$ and $A = 0.25$,

$$\frac{K_1(\text{comp.})}{K_1(\text{uncomp.})} = \frac{6}{3+(0.25)(6)(0.6)} = 1.54$$

then

$$K_1(\text{comp.}) = K_1(\text{uncomp.})\,(1.54)$$

$$= 0.6\,(1.54)$$

$$= 0.92$$

**Table 5.3 SUMMARY OF DESIGNS**

| Compensator | Transmittance | Ramp Error Coefficient | Closed-Loop Zeros | Closed-Loop Poles | Dominant Roots $\omega_n$ | Dominant Roots $\zeta$ |
|---|---|---|---|---|---|---|
| Uncompensated | $K = 3$ | 0.6 | None | $-0.767 \pm j0.793$, $-2.47$ | 1.10 | 0.70 |
| Cascade integral | $G_c(s) = \dfrac{K}{s}$ | $\infty$ | | Unstable | | |
| Cascade integral plus proportional | $G_c(s) = \dfrac{3(s+0.1)}{s}$ | $\infty$ | $-0.1$ | $-0.123$, $-0.724 \pm j0.694$, $-2.43$ | 1.00 | 0.72 |
| Cascade lag | $G_c(s) = \dfrac{3(s+0.1)}{s+0.01}$ | 6 | $-0.1$ | $-0.120$, $-0.729 \pm j0.705$, $-2.43$ | 1.01 | 0.72 |
| Cascade lead | $G_c(s) = \dfrac{60(s+1.6)}{s+16}$ | 1.2 | $-1.6$ | $-1.11$, $-1.32 \pm j1.9$, $-16.3$ | 2.31<br>1.11 | 0.57<br>1.00 |
| Cascade lag-lead | $G_c(s) = \dfrac{60(s+0.1)}{s+0.01} \times \dfrac{s+1.6}{s+16}$ | 12 | $-0.1$, $-1.6$ | $-0.109$, $-1.06$, $-1.29 \pm j1.86$, $-16.3$ | 2.26<br>1.06 | 0.57<br>1.00 |
| Feedback rate | $H_c(s) = \frac{1}{4}(s+4)$<br>$K = 6$ | 0.92 | None | $-0.845 \pm j1.37$, $-2.31$ | 1.61 | 0.52 |
| PID | $G_c(s) = \dfrac{4(s+1.25)^2}{s}$ | $\infty$ | $-1.5$, $-1.5$ | $-1 \pm j1.22$<br>$-1 \pm j1.22$ | 1.58 | 0.63 |