TRANSPORT OF SOLUTES
ADVECTION-DISPERSION EQUATION

V.6

Excerpts from Lecture Notes
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CHAPTER 2

APPLICATIONS OF THE CONSERVATION OF MASS EQUATION IN TURBULENT FLOW

2.1 Introduction

The literature of fluid mechanics and mass transfer contains many excellent discussions of the fundamental limitations involved in treating turbulent diffusion by the Fickian analogy with molecular diffusion; for example, Batchelor and Townsend (1956), Deacon (1959), Corrsin (1961), Neumann and Pierson (1966) and Csanady (1973). It is generally recognized that the most promising approach is by means of the statistical theory of turbulence which requires some form of Lagrangian description of the turbulent structure. That is, a treatment in terms of the erratic paths of fluid particles through the field of flow. This has been carried out for stationary, isotropic turbulence fields, but the vast amount of spatial and temporal turbulence measurements needed to perform the statistical correlations are a major hurdle. Turbulence associated with rivers, estuaries and the ocean is rarely stationary or isotropic and it must be concluded that the statistical approach has not as yet provided mathematical models for diffusion problems that are fundamentally less empirical than those based on the Fickian approach.

Within the area of application of the classical convective diffusion equation there are a great variety of mathematical models in terms of the coordinate systems to be employed, the number of dimensions to be considered in any given coordinate system and the method of specifying boundary and initial conditions. No attempt will be made to provide a complete catalog, therefore

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Batchelor, G.K. and Townsend, A.A., "Turbulent Diffusion", Surveys in Mechanics (Ed. Batchelor & Davies), Cambridge University Press, 1956.
Deacon, E.L., "The problem of Atmospheric Diffusion", Int. J. Air Poll., Vol. 2, pp. 92-108, Pergamon Press, 1959
Corrsin, S., "Turbulent Flow", American Scientist, Vol. 49, No. 3, pp. 300-325, 1961
Neumann, G. and Pierson, W.J., Prin. of Physical Oceanography, pp. 408-412, Prentice-Hall, 1966
Csanady, G.T., Turbulent Diffusion in the Environment, Reidel Publ. Co., 1973

we shall restrict our attention to the types of diffusion problems arising in air and water quality applications.

2.1.1 Initial Conditions

Problems in which the transport of substance is a function of time require the specification of an initial condition. Thus, the concentration throughout the spatial region must be given at the instant which is taken as the origin of the time coordinate t. The initial concentration distribution may be either a continuous or discontinuous distribution. As an example of the continuous type, it may be specified that the concentration at $t = 0$ is equal to zero (or a constant reference value) throughout the spatial region. A "delta function" is frequently employed for the discontinuous type of initial condition. In this case the concentration at $t = 0$ may be infinite at a point, along a line or on a plane and zero at all other places in space. An example is the "instantaneous" injection of a finite amount of substance at a fixed point.

2.1.2 Boundary Conditions

Boundary conditions may be specified at infinity or within a finite spatial region. The simplest condition, although not always the most physically meaningful, is the specification of the concentration at a boundary as a constant, or a function of time. A second type of boundary condition involves a prescribed flux of substance in which the concentration gradient is specified. For example, at an impervious surface, $\partial c/\partial n = 0$, at all points along the boundary where n is the direction normal to the surface.

2.1.3 Mathematical Techniques and Physical Interpretations

The convective diffusion equation (e.g., Eq. 1-30) is a parabolic equation of the second order and as such it is formally equivalent to the heat conduction equation. Many of the fundamental mathematical techniques were developed in the field of heat transfer and the most extensive accounts of methods of solving the Fickian equations are given by Crank (1956) and by Carslaw and Jaeger (1959).

Carslaw, H.S. and Jaeger, J.C., Conduction of Heat in Solids, Oxford, 2nd Ed., 1959, (later references to this will be designated as C. & J.)
Crank, J., The Mathematics of Diffusion, Oxford University Press, 1956

Appropriate solutions will be referred to where necessary without emphasis on the formal techniques employed for the solution. The following sections are primarily concerned with diffusion in forced convection; that is, the motion of the fluid, which transports the substance of interest, is governed by an applied pressure gradient or by gravity in the case of free surface flow. This means that the velocity field is specified independently and is not coupled with the concentration distribution. Examples in which this is not the case include: (1) free or natural convective motions in a gravitational field in which the fluid motion is maintained by differences in density and (2) jet diffusion in which the fluid motion is generated by the injection of fluid into a system which is otherwise at rest.

One fundamental approach to transport processes of interest to air and water quality control is by the method of sources. In this manner concentration distributions for the instantaneous or continuous introduction of a pollutant can be developed for both unsteady and steady state conditions. The solutions will be developed in terms of the dimensionless concentration variable, c, (mass of substance per unit mass of solution or weight of substance per unit weight of solution). In specific air and water quality applications c may refer to the concentration of: radioactive material, SO_2, dye, salinity, chlorinity, biochemical oxygen demand, dissolved oxygen or any other tracer substance. It will also be assumed that if the substance is non-conservative it decays in accordance with a first order reaction in which,

$$\frac{r_A}{\rho} = -Kc \qquad (2\text{-}1)$$

Equation (2-1) states that the substance disappears at a rate which is linearly proportional to the remaining concentration.

In turbulent motion, the turbulent diffusion coefficients are many orders of magnitude larger than the molecular diffusion coefficients. Hence, unless we are concerned with diffusion near a solid boundary (where the turbulence is damped), usually it is permissible to neglect the molecular diffusion terms. Under the above assumptions, the convective diffusion equation for turbulent flow (Eq. 1-30) may be written (omitting the bars and subscripts on the concentration and velocity terms),

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}\left(E_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(E_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(E_z \frac{\partial c}{\partial z}\right) - Kc \qquad (2\text{-}2)$$

In the majority of concentration distribution problems involving turbulent flow, the convective transport terms on the left-hand side of Equation (2-2) are by far the most important and yet have received the least attention. This is partly due to the carry-over of analytical techniques from the field of heat conduction which is primarily concerned with temperature distribution in solid bodies. Such bodies are either at rest ($u = v = w = 0$) or they are moving in a steady, uniform motion (e.g., $u = U$ = constant, $v = w = 0$) along one axis of the coordinate system. In a turbulent fluid system the velocity field may be steady, but it is seldom uniform spatially. In fact the terms "steady" and "uniform" must be carefully defined because the magnitude of the diffusion terms on the right side of Equation (2-2) depend critically upon the method of averaging velocities in space and time. In a three-dimensional turbulent flow each velocity component, which appears as a coefficient of a concentration gradient in Equation (2-2), is in general a function of space and time. Thus,

$$\begin{aligned} u &= f_1(x,y,z,t) \\ v &= f_2(x,y,z,t) \qquad (2\text{-}3) \\ w &= f_3(x,y,z,t) \end{aligned}$$

and in order to find $c = f_4(x,y,z,t)$ the functional relations of Equations (2-3) must be specified. Formally this would require four additional differential equations (three components of the equations of motion and the continuity equation for the fluid as a whole). These difficulties have usually been avoided by gross simplifications of the velocity field with the result that the turbulent diffusion terms on the right-hand side of Equation (2-2) are forced to account for the assumptions regarding the velocity distribution. In certain cases there is an analytical basis for shifting some aspects of the convective transport into the conductive or diffusive terms. It then becomes important to distinguish between diffusion (usually associated with products such as $\overline{u'c'}$) and "dispersion" which is associated with an approximate specification of the velocity field.

Analytical solutions of Equation (2-2), such as those presented in the following sections, are possible only for simple velocity fields. Thus we will

sometimes refer to the coefficients on the right-hand side as dispersion coefficients in order to emphasize their dependence on the assumed velocity field. If the diffusion or dispersion coefficients are assumed to be homogeneous and anisotropic, Equation (2-2) becomes

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} + E_z \frac{\partial^2 c}{\partial z^2} - Kc \qquad (2\text{-}4)$$

If they are both homogeneous and isotropic, Equation (2-4) can be written as

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = E\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) - Kc \qquad (2\text{-}5)$$

The equation for molecular diffusion may be recovered by setting $E = D$ where D is the molecular diffusion coefficient.

The use of numerical techniques in the solution of mass transport equations obviously holds much promise. Nevertheless, the analytical solutions are helpful in terms of a basic understanding and in many cases they are a useful first approximation to actual problems. They are also of importance in providing a check on the accuracy of numerical techniques.

2.2 Instantaneous Point Source

We consider a group of problems in which the concentration is a function of x, y, z and t. The flow field is of infinite or semi-infinite extent or is bounded by parallel planes. A substance is assumed to be instantaneously injected into a fluid at a point. The substance is non-conservative and is assumed to decay at a rate proportional to the concentration, i.e. "first-order decay".

The instantaneous point source, located at x_1, y_1, z_1, is generated at $t = 0$ in an infinite fluid. The x axis is aligned with the direction of fluid flow which is characterized by the velocity vector $u = U$ = constant. The diffusivity is homogeneous and anisotropic and has principal components in the x,y,z direction. Equation (2-4) becomes

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} = E_x\frac{\partial^2 c}{\partial x^2} + E_y\frac{\partial^2 c}{\partial y^2} + E_z\frac{\partial^2 c}{\partial z^2} - Kc \qquad (2\text{-}6)$$

INSTANTANEOUS POINT SOURCE EQUATION

The advective term and the decay term can be eliminated by the following change of variables,

$$c = \phi e^{-Kt}$$

$$x = X+Ut$$

← Moving field, eliminates $U\frac{\partial c}{\partial x}$ term

and Equation (2-6) can be written as

$$\frac{\partial \phi}{\partial t} = E_x\frac{\partial^2 \phi}{\partial X^2} + E_y\frac{\partial^2 \phi}{\partial y^2} + E_z\frac{\partial^2 \phi}{\partial z^2} \qquad (2\text{-}7)$$

The solution to Equation (2-7) for a delta function input at x_1, y_1, z_1, is given by C & J (Equation 8, pg. 257); in terms of the original variables this becomes

$$c = \frac{M}{\rho(4\pi t)^{3/2}(E_xE_yE_z)^{1/2}} \exp - \left\{ \frac{[(x-x_1)-Ut]^2}{4E_xt} + \frac{(y-y_1)^2}{4E_yt} + \frac{(z-z_1)^2}{4E_zt} + Kt \right\} \qquad (2\text{-}8)$$

and it is assumed that the diffusivities E_x, E_y and E_z are independent of time. As $t \to 0$ this expression tends to zero at all points, except at the location of the source x_1, y_1, z_1 where the concentration becomes infinite. Also, as $t \to \infty$ the concentration tends to zero everywhere. In Equation (2-8), M represents the total amount of mass of substance introduced and ρ is the density of the mixture of substance and fluid which may be taken as a constant equal to the density of receiving fluid. Note that if the numerator and denominator of the equation are multiplied by g (the acceleration of gravity), M may be replaced by W, the weight of substance introduced, and ρ may be replaced by γ, the specific weight of the receiving fluid.

Usually it is convenient to let the origin of the coordinate system coincide with the location of the point source; therefore, x_1, y_1 and z_1 are equal to zero. Special cases of Equation (2-8) include an isotropic turbulent flow field in which $E_x = E_y = E_z = E$ and the molecular diffusion ($E = D$) of a

conservative substance (K = 0) in a fluid at rest (U = 0), in the latter situation Equation (2-8) can be written as

$$c = \frac{M}{\rho(4\pi Dt)^{3/2}} \exp - \left[\frac{r^2}{4Dt}\right] \qquad (2\text{-}9)$$

IN RADIAL COORDINATES

where

$$r = \sqrt{x^2+y^2+z^2} \quad . \quad (x_1 = y_1 = z_1 = 0)$$

2.2.1 Effect of a Finite Domain

This is a convenient point at which to introduce the method of "images" as a mathematical technique for dealing with receiving bodies of fluids of finite extent. The simplest case is one in which the receiving fluid is semi-infinite with a plane, zero-flux, boundary at the z = o axis as shown in Figure 2.1.

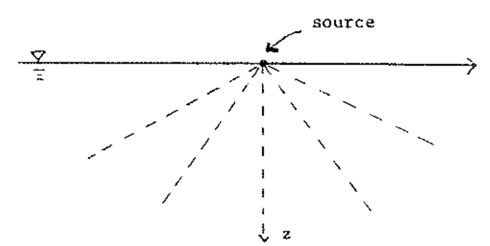


FIGURE 2.1 Source at the free surface of a semi-infinite fluid.

This would arise in the case of a source located at the free surface of the ocean or (upon inverting the figure) for a source at ground level in the ocean or in the atmosphere.

The application of Equation (2-8) would allow the transport of one-half of the injected mass above the free surface, obviously a physical impossibility if we prescribe that there can be no flux $(\frac{\partial c}{\partial z} = 0)$ across this boundary.

One method of handling this situation is to superimpose an identical injection at the same point resulting in the total transport of mass below the surface equal to the total transport of mass above and below the reference line in the case of an infinite fluid. The resulting concentrations will be twice the values given by Equation (2-8). Similarly, if the source is located some distance z below the surface, the required placement of the necessary "image" source would be at z above the surface. This method provides for the addition of the amount of mass across the boundary exactly equal to the amount of mass which the equation derived for an infinite fluid would allow to be lost across the boundary. As shown in Figure 2-2, the image source and the real source must be symmetrical about the boundary, thereby allowing the mass flux from the image to be equal to the mass flux from the real source.

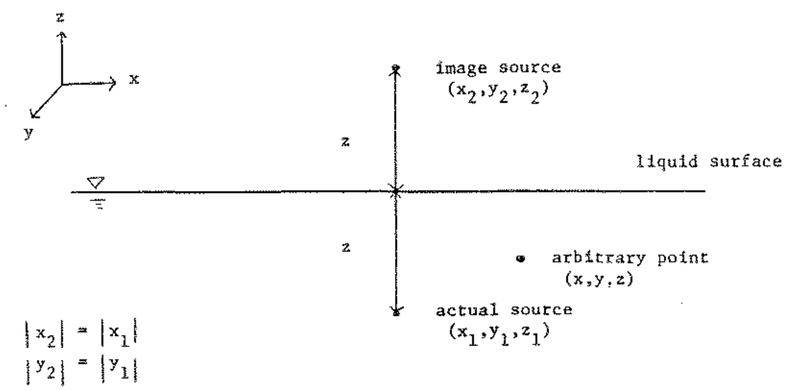


FIGURE 2.2 Source and image in a semi-infinite fluid.

In terms of Equation (2-8), the solution for the concentration at an arbitrary point (x,y,z) with a source located at (x_1,y_1,z_1) in a semi-infinite fluid (and its image at x_2,y_2,z_2) is given by

$$c = \frac{M}{\rho(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \exp - \overbrace{\left\{ \frac{(x-x_1)^2}{4E_x t} + \frac{(y-y_1)^2}{4E_y t} + \frac{(z-z_1)^2}{4E_z t} \right\}}^{I}$$

$$+ \frac{M}{\rho(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \exp - \underbrace{\left\{ \frac{(x-x_2)^2}{4E_x t} + \frac{(y-y_2)^2}{4E_y t} + \frac{(z-z_2)^2}{4E_z t} \right\}}_{II}$$

or:

$$c = \frac{M}{\rho(4\pi t)^{3/2}(E_x E_y E_z)^{1/2}} \cdot \left\{ \exp(-I) + \exp(-II) \right\} \qquad (2\text{-}10)$$

A further extension of this concept is its application to a wide (i.e. two-dimensional) waterway with a channel bed and a constant finite depth, H. Consider a source at some point above the bottom as shown in Figure 2-3. (For convenience, the point source is located at H/2).

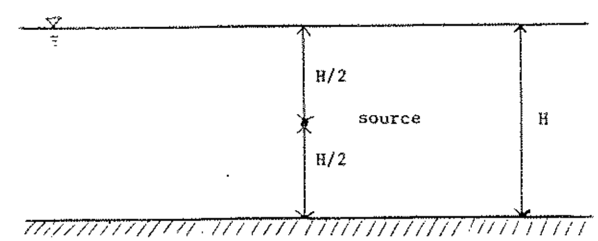


FIGURE 2.3 Point source in a channel of uniform depth.

In this case two zero flux boundaries must be accounted for, which theoretically requires the placement of an infinite series of double images. An image is required each time the solution derived for an infinite fluid (Eq. 2-8) allows the transport of mass across a boundary. The placement of

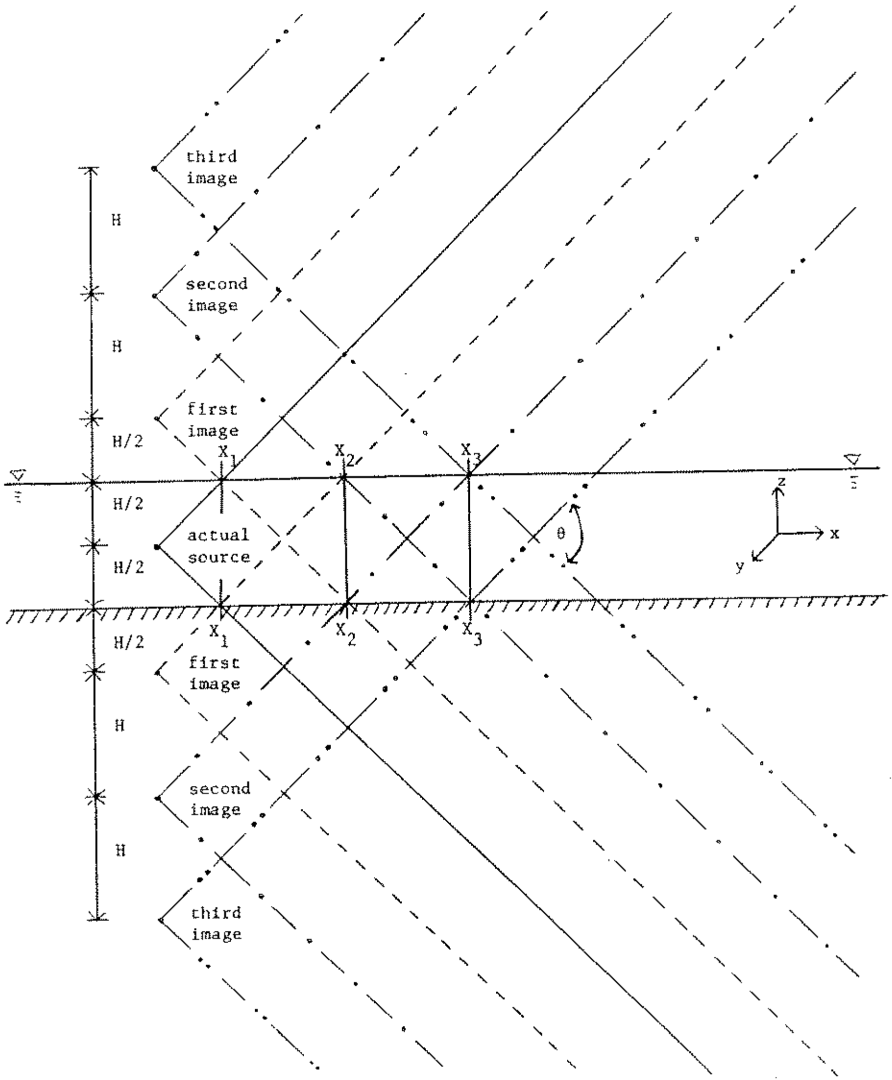


FIGURE 2.4 Images for a point source in a channel of uniform depth.

the image sources is shown in Figure 2·4. For graphical expediency, the diffusion from each source is shown to be linear at an angle θ, and is symmetrical about a line parallel with the x axis.

For the actual source, it is seen that the solution for an infinite fluid allows mass to cross both the top and bottom boundaries at X_1. (The points are equal in x because the source is placed at the midpoint of the depth H). Therefore, it is required that mass be added from these points forward at a rate at which it is being allowed to leave. This is accomplished by placement of images at H/2 above and below the boundaries. Again it is seen that mass added by these images is allowed to leave at X_2 requiring the addition of second images at H above and below the first images. Similarly, the situation arises again at X_3 which requires the placement of a third set of images at H above and below the second images. This process continues out to infinity.

Generally, only two sets of images need be considered as the contribution of the higher images to the concentration in the waterway is small. However, each case is specific and must be investigated for its sensivitity to the number of images required particularly when calculating concentrations at large distances from the source.

The effect of both a finite depth and width of the flow field can be treated mathematically either by the method of images or by integral transforms (Cleary and Adrian, 1973). The coordinate system associated with a rectangular, parallel wall channel of uniform width B and depth H is shown in Figure 2.5.

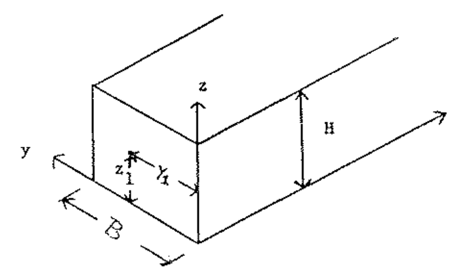


FIGURE 2.5 Point source in a channel of uniform width and depth.

Cleary, R.W. and Adrian, D.D. "New Analytical Solutions for Dye Diffusion Equations", Proc. A.S.C.E., Vol. 99, No. EE3, June 1973

The point source is located at $x = 0$, $y = y_1$ and $z = z_1$ and the flow field is described by a constant velocity $u = U$. The solution to Eq. (2-6), with the boundary conditions of zero concentration gradients on all boundaries, is in the form of a double infinite series. The solution is exact as posed mathematically; however, it is not very realistic from a physical standpoint. A real fluid flowing in a rectangular channel, with or without a free surface, will have velocity gradients in the lateral and vertical directions due to boundary shear. The effect of velocity shear on the mixing process is properly accounted for by expressing the spatial variation of the velocity in the advective terms on the left-hand side of Eq. (2-4). If the velocity variation is ignored in favor of the cross-sectional mean velocity U, as in Eq. (2-6), the mixing due to shear must be reflected by increasing magnitudes of the diffusion coefficients, in which case they should be called dispersion coefficients. In any event, it is not correct to assume that the magnitudes of E_x, E_y and E_z are spatially constant. Thus the basic premise of a homogeneous turbulent flow is invalid. The subject of dispersion will be treated quantitatively in a later section. An example of turbulent diffusion in a simple shear flow is discussed in the following paragraph.

2.5 Continuous Point Source

An important group of mass transport problems is concerned with the distribution of concentration from a source which is injecting a substance in a continuous manner. Solutions for continuously discharging sources can be built up by summing the effect of closely spaced (in time) instantaneous injections. In this case the summing or integration is with respect to time whereas in the previous sections the integration of sources was done in the spatial dimensions. The procedure is illustrated in Fig. 2.13 which shows schematically the concentration pulses produced by a series of instantaneous injections consisting of equal amounts of mass dM,

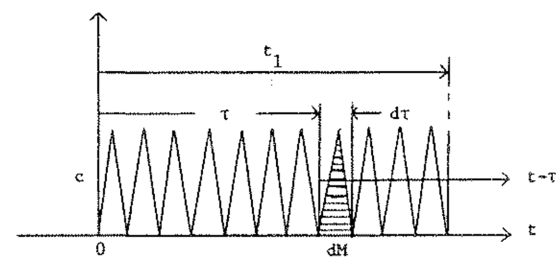


Figure 2.13 Distribution of instantaneous sources forming a continuous source.

Time, t, is measured from the start of the series of injections and the duration of the series is t_1. We may single out one instantaneous injection occurring in the time interval between τ and $\tau + d\tau$ during which an amount of substance dM is injected.

The concentration response dc due to this injection depends on whether the source is a point, line or plane. In any case it can be obtained from one of the instantaneous solutions in the previous sections by replacing M by dM, c by dc and t by the appropriate time variable $t - \tau$, since the time in the previous solutions is always measured from the time of the instantaneous injection.

As an example of the method we consider a continuous point source in a uniform flow field of infinite extent with homogeneous, non-isotropic turbulence. The point source is located at x_1, y_1, z_1. The concentration response dc is given by Eq. (2-8)

$$dc = \frac{dM}{\rho[4\pi(t-\tau)]^{3/2}(E_x E_y E_z)^{1/2}} \exp-\left[\frac{[(x-x_1) - U(t-\tau)]^2}{4E_x(t-\tau)} + \frac{(y-y_1)^2}{4E_y(t-\tau)} + \frac{(z-z_1)^2}{4E_z(t-\tau)} + K(t-\tau)\right] \quad (2\text{-}36)$$

The time rate of injection of mass is given by

$$q = \frac{dM}{d\tau} \quad (2\text{-}37)$$

In Eq. (2-36) we can replace dM by $q\,d\tau$ and integrate with respect to time for the duration of the injection from $\tau = 0$ to $\tau = t_1$, thus

$$c = \int_0^{t_1} \frac{q}{\rho[4\pi(t-\tau)]^{3/2}(E_x E_y E_t)^{1/2}} \exp-\left[\frac{[(x-x_1) - U(t-\tau)]^2}{4E_x(t-\tau)} + \frac{(y-y_1)^2}{4E_y(t-\tau)} + \frac{(z-z_1)^2}{4E_z(t-\tau)} + K(t-\tau)\right] d\tau \quad (2\text{-}38)$$

Equation (2-38) may be simplified by the following substitutions,

$$a = \frac{(x-x_1)^2}{4E_x} + \frac{(y-y_1)^2}{4E_y} + \frac{(z-z_1)^2}{4E_z}$$

$$b = \frac{U^2}{4E_x} + K$$

$$d = \frac{q \exp\left[\frac{(x-x_1)U}{2E_x}\right]}{\rho(4\pi)^{3/2}(E_x E_y E_z)^{1/2}}$$

in which case,

$$c = d\int_0^{t_1} \frac{1}{(t-\tau)^{3/2}} \exp-\left[\frac{a}{(t-\tau)} + b(t-\tau)\right] d\tau \quad (2\text{-}39)$$

By defining a new variable $\zeta = \sqrt{\frac{1}{(t-\tau)}}$, Eq (2-39) becomes

$$c = 2d\int_{\sqrt{1/t}}^{\sqrt{\frac{1}{t-t_1}}} \exp-\left(a\zeta^2 + \frac{b}{\zeta^2}\right) d\zeta \quad (2\text{-}40)$$

The solution to this integral is

$$c = \frac{d\sqrt{\pi}}{2\sqrt{a}}\left\{e^{2\sqrt{ab}}\left[\mathrm{erf}\left(\sqrt{\frac{a}{(t-t_1)}} + \sqrt{b(t-t_1)}\right) - \mathrm{erf}\left(\sqrt{\frac{a}{t}} + \sqrt{bt}\right)\right] + e^{-2\sqrt{ab}}\left[\mathrm{erf}\left(\sqrt{\frac{a}{(t-t_1)}} - \sqrt{b(t-t_1)}\right) - \mathrm{erf}\left(\sqrt{\frac{a}{t}} - \sqrt{bt}\right)\right]\right\} \quad (2\text{-}41)$$

Equation (2-41) gives the concentration distribution at times $t > t_1$; that is, at times subsequent to a continuous injection of duration t_1. The special case of a continuous injection isgiven by setting $t_1 = t$; thus, Eq. (2-41) becomes

$$c = \frac{d\sqrt{\pi}}{2\sqrt{a}}\left\{e^{2\sqrt{ab}}\,\mathrm{erfc}\left[\left(\sqrt{\frac{a}{t}} + \sqrt{bt}\right)\right] + e^{-2\sqrt{ab}}\left[\mathrm{erfc}\left(\sqrt{\frac{a}{t}} - \sqrt{bt}\right)\right]\right\} \quad (2\text{-}42)$$

In many situations involving continuous injection it is sufficient to have the steady state solution, this may be found from Eq. (2-42) by letting $t \longrightarrow \infty$.

The steady state solution is

$$\bar{c} = \frac{d\sqrt{\pi}}{\sqrt{a}} \exp - [2\sqrt{ab}] \qquad (2\text{-}43)$$

EXACT SOLN. An approx. soln. is given just ahead by neglecting the $\frac{\partial^2 c}{\partial x^2}$ term.

Equation (2-43) is a solution of the steady state mass transport equation for a continuous injection from a point source located at x_1, y_1, z_1:

$$U\frac{\partial c}{\partial x} = E_x \frac{\partial^2 c}{\partial x^2} + E_y \frac{\partial^2 c}{\partial y^2} + E_z \frac{\partial^2 c}{\partial z^2} - Kc \qquad (2\text{-}44)$$

As a special case of Eq. (2-43), consider the steady state solution for a point source at $x_1 = y_1 = z_1 = 0$ in a homogeneous, isotropic turbulent field in which $E_x = E_y = E_z = E$. Thus,

$$\bar{c} = \frac{q}{4\pi\rho E r} \exp - \left[\frac{r\sqrt{U^2+4EK} - xU}{2E}\right] \qquad (2\text{-}45)$$

where, $r = \sqrt{x^2+y^2+z^2}$

As $r \rightarrow 0$, $\bar{c} \rightarrow \infty$, a physically impossible condition. The problem is resolved by considering that q (the mass of tracer substance emitted per unit time) may be replaced by

$$q = \bar{c}_i \rho Q \qquad (2\text{-}46)$$

where, $\bar{c}_i$ = concentration of the tracer in the injected mixture,

Q = volume rate of flow of the point source

Equation (2-45) may be rewritten as

$$\frac{\bar{c}}{\bar{c}_i} = \frac{Q}{4\pi E r} \exp - \left[\frac{r\sqrt{U^2+4EK} - xU}{2E}\right] \qquad (2\text{-}47)$$

and the solution is invalid for small values of r such that $\bar{c}/\bar{c}_i > 1$. It should be noted that the point source is assumed to have no momentum relative to the

receiving fluid. This is consistent with the basic assumption of "passive diffusion".

If the source is located on the boundary of a semi-infinite body (ground, or water surface) the concentrations will be twice the values given by the above equations for the same rate of injection. Sutton (1953) considered the effect of a nearby boundary on an elevated source (a chimney stack) by the method of images. In the absence of decay (K=0), the peak concentration (y = z = 0) varies as x^{-1} in Eq. (2-47). Meteorological observations of point sources emitting from a ground surface indicate a more rapid decrease in the order of $x^{-1.75}$ (Sutton, 1953). Therefore, the above equations are at best a crude approximation because of the assumption of constant advective velocity and isotropic diffusion. Kuo (1976) gives analytical solutions for a continuous point source in a uniform rectangular channel (see Fig. 2.5) with a uniform (U = constant) flow field.

Sutton, O.G., Micrometeorology, McGraw-Hill. p. 139, 1953.

Kuo, E.Y.T., "Analytical Solution for 3-D Diffusion Model", Proc. ASCE, Vol. 102, No. EE4, August 1976

2.5.1 Approximate Solution for a Steady State Continuous Point Source

An approximate form of the steady state solution for a continuous point source in non-isotropic turbulence (Eq. 2-43) may be readily developed. The approximation, analogous to the "boundary layer approximation" in fluid mechanics, is frequently employed in writing the governing differential equation for steady state mass transport problems. The continuously discharging point source is located at $x_1 = y_1 = z_1 = 0$. The parameter, a, in Eq. (2-43) can be rewritten as

$$a = \frac{x^2}{4E_x}\left[1 + \left(\frac{y^2E_x}{x^2E_y} + \frac{z^2E_x}{x^2E_z}\right)\right] \qquad (2\text{-}48)$$

If the quantity,

$$a_1 = \frac{y^2E_x}{x^2E_y} + \frac{z^2E_x}{x^2E_z} << 1$$

(as is the case as x becomes large), an approximate form for $\sqrt{a}$ may be obtained by means of the binomial expansion, thus

$$\sqrt{a} \cong \frac{x}{2\sqrt{E_x}}\left(1 + \frac{a_1}{2}\right) \qquad (2\text{-}49)$$

In a similar manner,

$$b = \frac{U^2}{4E_x}\left[1 + \frac{4E_xK}{U^2}\right]$$

and if,

$$b_1 = \frac{4E_xK}{U^2} << 1 \qquad (2\text{-}50)$$

it follows that the approximate form

$$\sqrt{b} \cong \frac{U}{2\sqrt{E_x}}\left(1 + \frac{b_1}{2}\right) \qquad (2\text{-}51)$$

Therefore,

$$\sqrt{ab} \cong \frac{xU}{4E_x}\left[1 + \frac{a_1}{2} + \frac{b_1}{2}\right] \qquad (2\text{-}52)$$

The approximation for the steady state concentration distribution is obtained by substituting Eqs. (2-49) and (2-52) into Eq. (2-43) and rearranging,

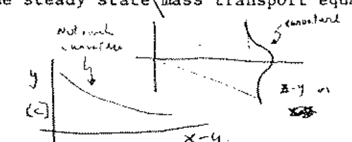
$$\bar{c} \cong \frac{q}{4\pi\rho(E_yE_z)^{1/2}x}\exp-\left[\frac{y^2U}{4xE_y} + \frac{z^2U}{4xE_z} + \frac{Kx}{U}\right] \qquad (2\text{-}53)$$

Note that the conditions for the approximation are given by Eqs. (2-48) and (2-50). The diffusivity, E_x, is no longer a parameter in the approximate solution. On physical grounds this is due to the fact that the "curvatures" of the concentration distribution in the lateral (y,z) directions are much greater than in the longitudinal (x) direction. Thus

$$\frac{\partial^2 c}{\partial x^2} << \frac{\partial^2 c}{\partial y^2} \quad \left(\text{and } \frac{\partial^2 c}{\partial z^2}\right)$$

Equation (2-53) is an exact solution of the steady state mass transport equation

$$U\frac{\partial c}{\partial x} = E_y\frac{\partial^2 c}{\partial y^2} + E_z\frac{\partial^2 c}{\partial z^2} - Kc \qquad (2\text{-}54)$$



Equation (2-54) is frequently employed as the basis for steady state mathematical models by means of analytical or numerical techniques. Examples are given in the following section.