SOLUTION OF DIFFUSION EQN.

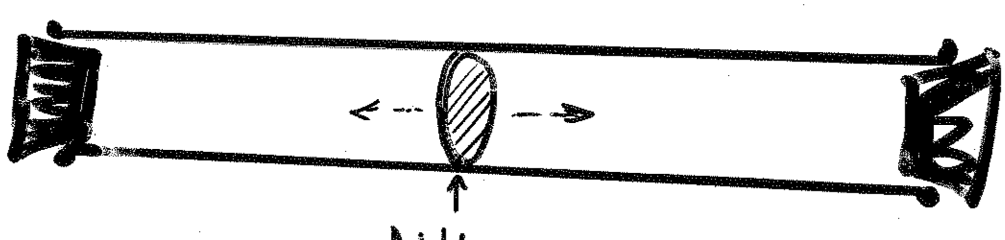
(CONS. OF MASS FOR A BINARY FLUID)

[NO ADVECTION]

$$\frac{\partial c}{\partial t} = D_{AB}\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right)$$

SECOND ORDER
BUT <u>LINEAR</u>

HENCE WE CAN "SUPERPOSE" SIMPLE SET-UPS TO SOLVE MORE COMPLEX ONES



NH_3

INFINITESIMALLY THIN DISK @

$x=0$

$t=0$

GOV. EQN. : 1-D in X

$$\frac{\partial C}{\partial t} = D_{AB} \frac{\partial^2 C}{\partial x^2} \qquad (8)$$

INIT. CONDS: $t=0$

$$\begin{cases} C \rightarrow \infty & \text{for } x \rightarrow 0 \\ C = 0 & \text{for } x \neq 0 \end{cases}$$

BOUNDARY COND. $C \rightarrow 0$ for $|x| \rightarrow \infty$

SOLVE W/ LAPLACE TRANSFORM

$$C = \frac{B}{\sqrt{t}} \exp - \left[\frac{x^2}{4Dt} \right] \qquad (9)$$

$t=0$? CONC., IN GENERAL, POORLY DEFINED

$x=0 \quad C \rightarrow \infty$

$x \neq 0 \quad C \rightarrow 0$

Not useful

Use cons. of mass. of total NH_3

$$M_{NH_3} = \int_{-\infty}^{\infty} CA\,dx = A\int_{-\infty}^{\infty} C\,dx \quad (10)$$

TOTAL MASS OF NH_3 IN WHOLE TUBE

AND, PREVIOUS SOLN. GIVES US $C(x)$ SO PLUG IN HERE

$$M_{NH_3} = A\int_{-\infty}^{+\infty} \boxed{\frac{B}{\sqrt{t}} \exp -\left[\frac{x^2}{4Dt}\right]}\, dx \quad (11)$$

FIND THIS SUBSTITUTION:

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \sqrt{\pi}/a \quad (12)$$

Let $a = 1/\sqrt{4Dt}$

$$M_{NH_3} = \frac{AB}{\sqrt{t}}\left[\frac{\sqrt{\pi}}{1/\sqrt{4Dt}}\right] = AB\,\frac{\sqrt{4\pi Dt}}{\sqrt{t}} = 2AB\sqrt{\pi D} \quad (13)$$

TIME INVARIANT MASS

Claim $\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \sqrt{\pi}/a$

Strategy: convert to a Normal pdf, the integral across which is 1.

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\left(\frac{1}{\sqrt{2}a}\right)^2}}\, dx$$

$$= \frac{\sqrt{2\pi}}{\sqrt{2}\,a} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\frac{\sqrt{2\pi}}{\sqrt{2}a}}\, e^{-\frac{1}{2}\left(\frac{x^2}{\left(\frac{1}{\sqrt{2}a}\right)^2}\right)}}_{N\left(0,\ \sigma^2 = \left(\frac{1}{\sqrt{2}a}\right)^2\right)}\, dx$$

$$= \sqrt{\pi}/a \quad \blacksquare$$

[COURTESY OF A STUDENT
WHO WAS A FORMER MATH MAJOR]

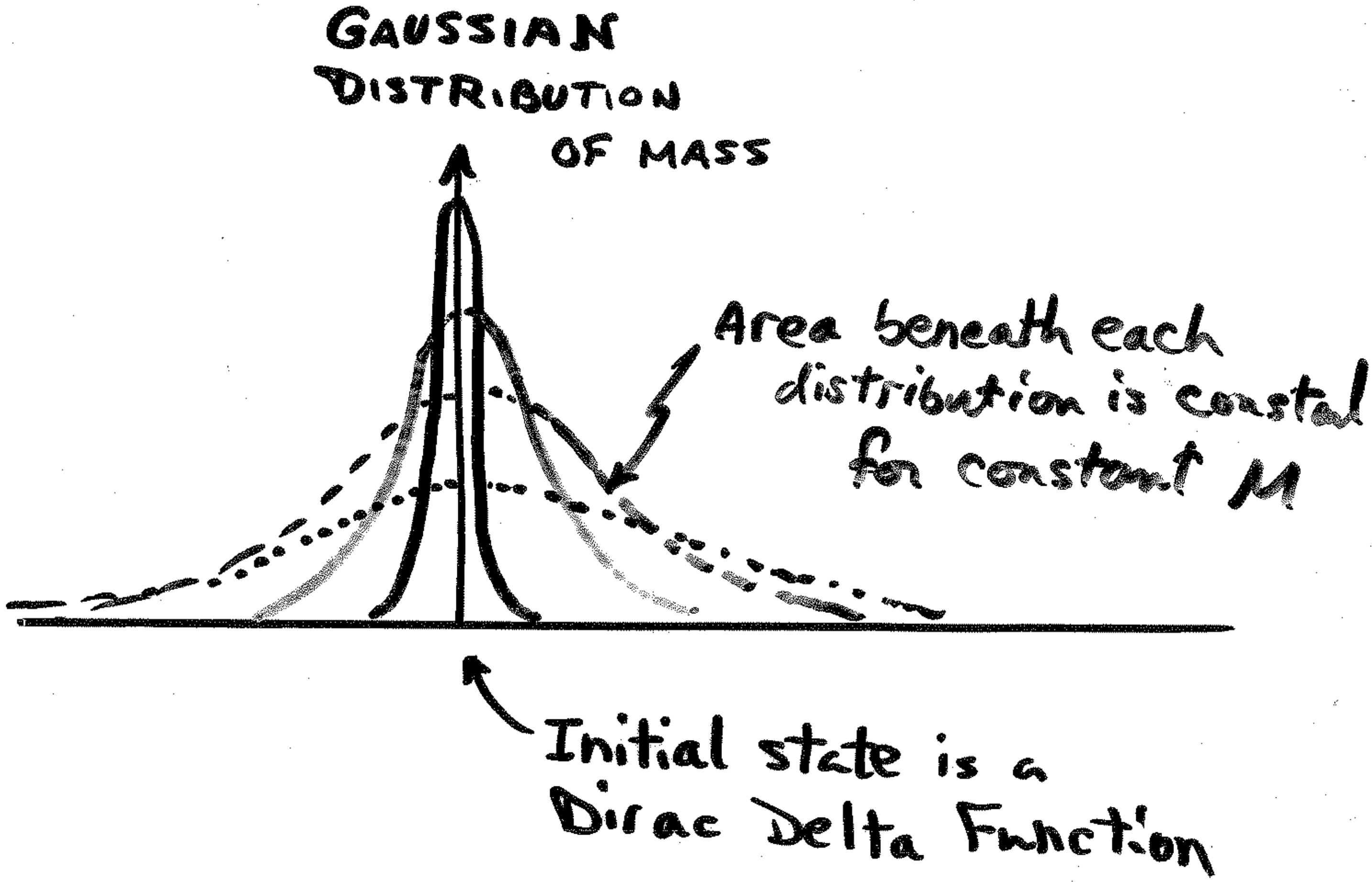
REARRANGE:

$$\mathcal{B} = M / 2A\sqrt{\pi D} \qquad (14)$$

$$C(x,t) = \frac{M_{NH_3}}{2A\sqrt{\pi D t}} \exp\left[-\frac{x^2}{4Dt}\right] \qquad (14) \downarrow (9) = (15)$$

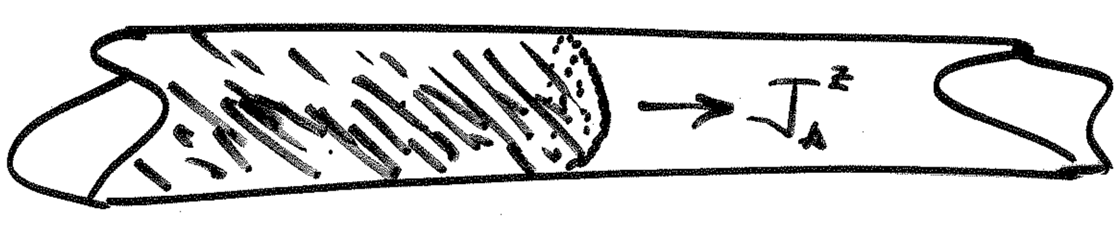
Which is the GAUSSIAN (or (NORMAL) DISTRIBUTION

Tells us the one-dimensional diffusion away from a finite "slug" of mass M



So, big deal. But the PRINCIPLE OF SUPERPOSITION allows us to use this elementary solution to build up more COMPLEX SYSTEMS.

(E.g., [redacted] NH_3 diffusion in channel...)



DIFFUSION / DISPERSION / ETC.

Diffusion: Transport via RANDOM MOLECULAR motions.

- Independent of location
- Irreversible, entropy-driven

Advection: "Passive" transport via bulk motions of fluid

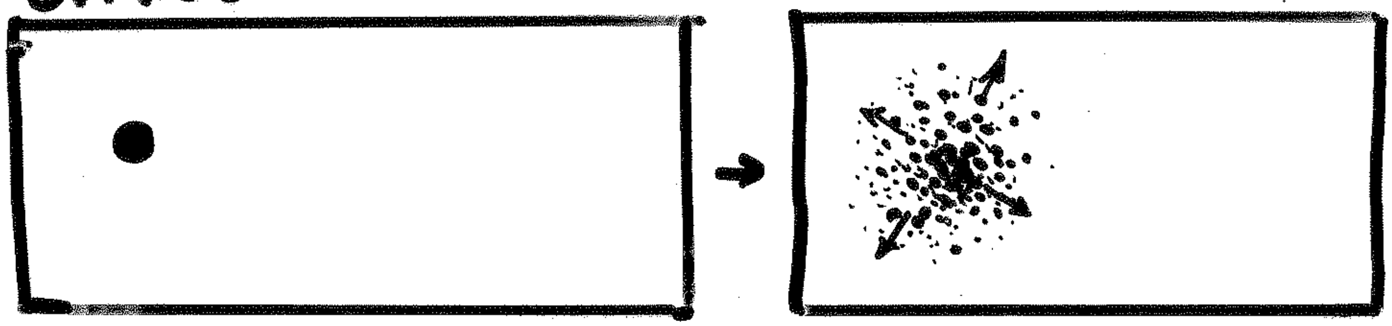
- Function of bulk fluid displacement (Lagrangian "tracer" of motions)

Turbulent Diffusion: Transport via chaotic motions of fluid turbulence.

- Depends on structure and intensity of turbulent motions.
- Function of location (and time if unsteady flow)

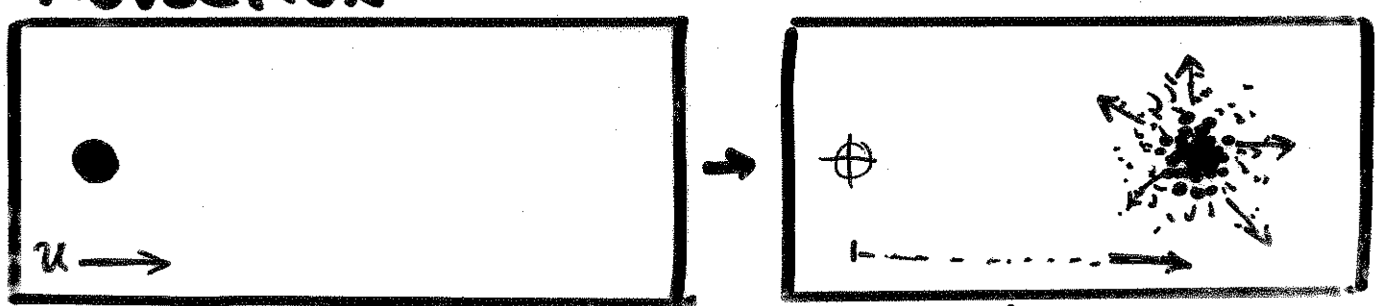
INITIAL TIME ⇩ ..... "LATER" TIME ⇩

DIFFUSION



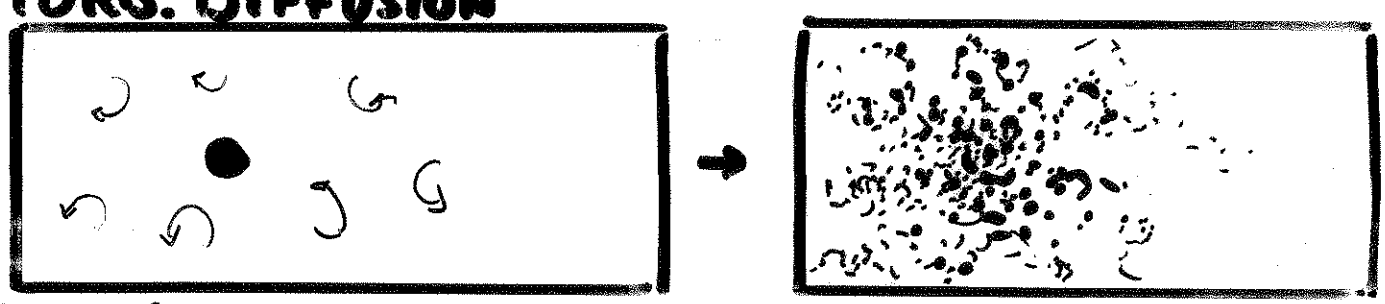
- No fluid motion
- Yet, dye moves "outward"

ADVECTION



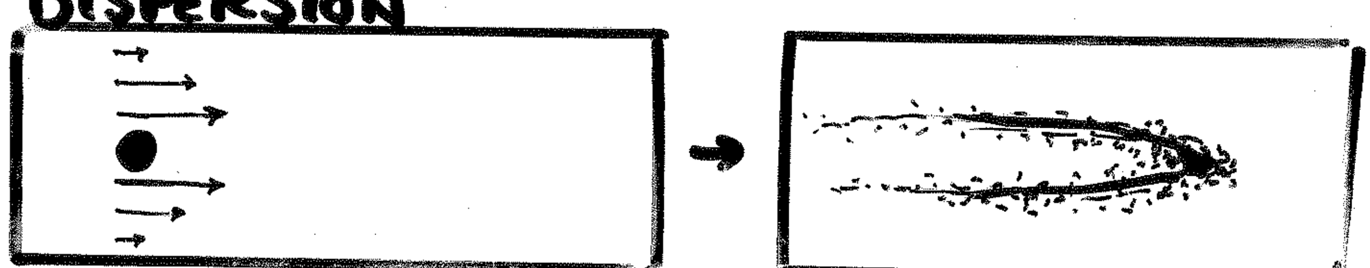
- UNIFORM VELOCITY
- Dye advects + diffuses

TURB. DIFFUSION

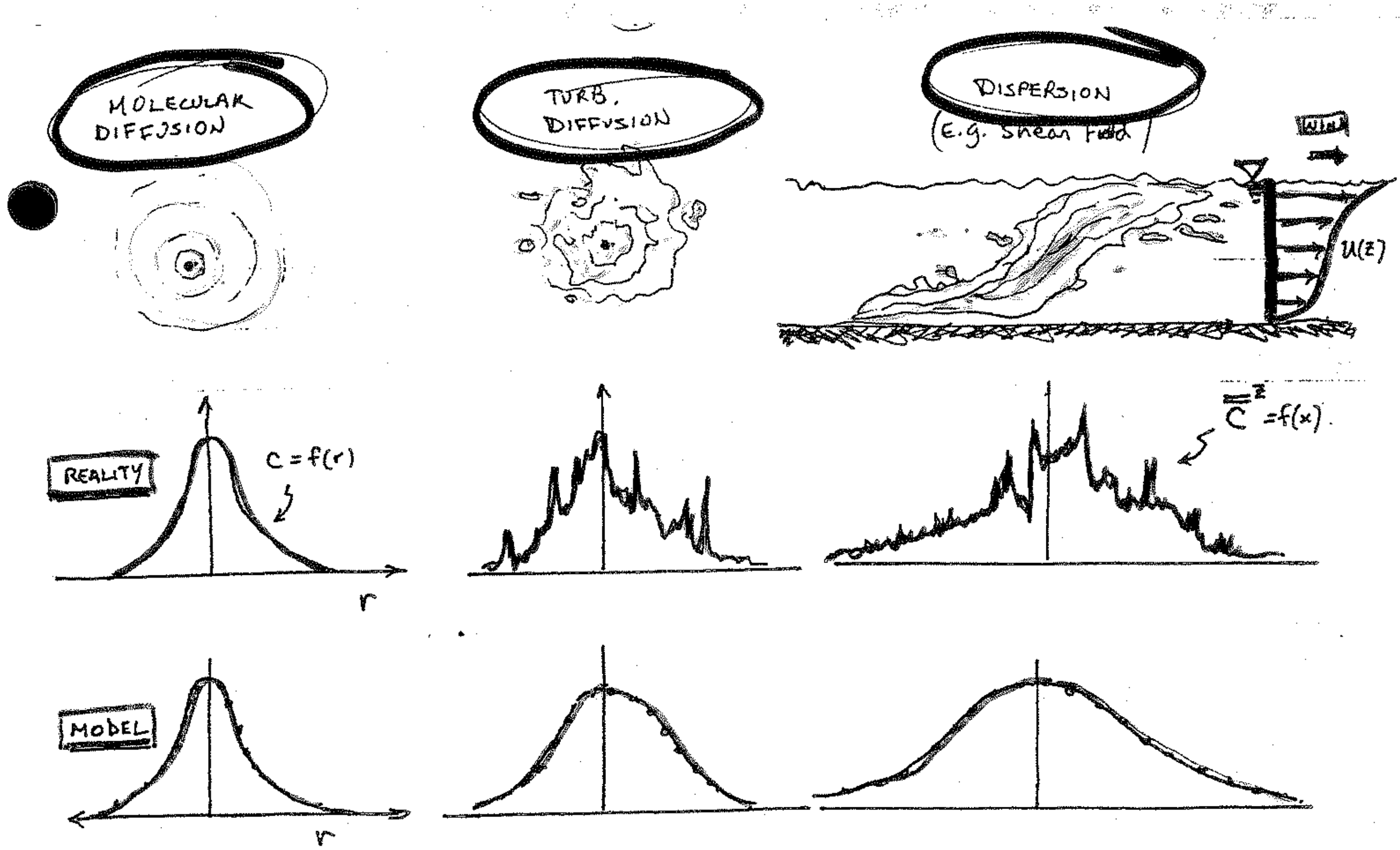


- No <u>net</u> motion
- Turbulent eddies
- Dye <u>QUICKLY</u> spreads outward

DISPERSION



- Nonuniform flow
- Shear-Induced spreading

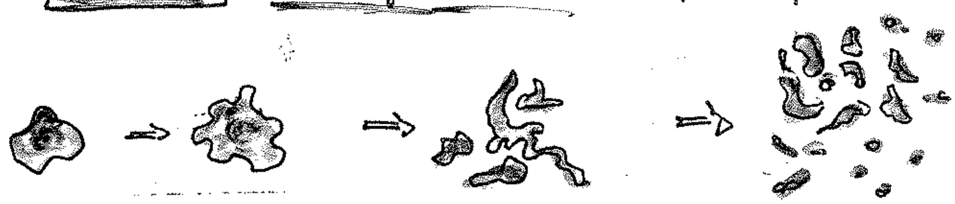


Compare "reality" with predictions by Fickian Analogy.

Or, consider an oil slick: How good is Fickian analogy?

STAGE ONE: Spreading by gravity

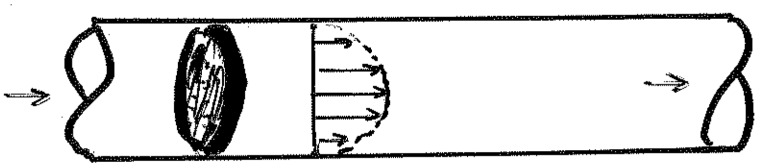
STAGE TWO Break up of slick into separate patches

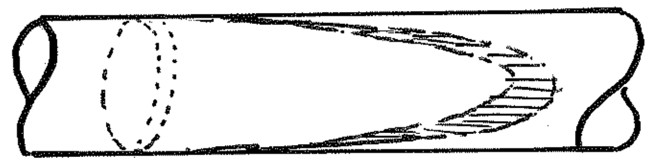


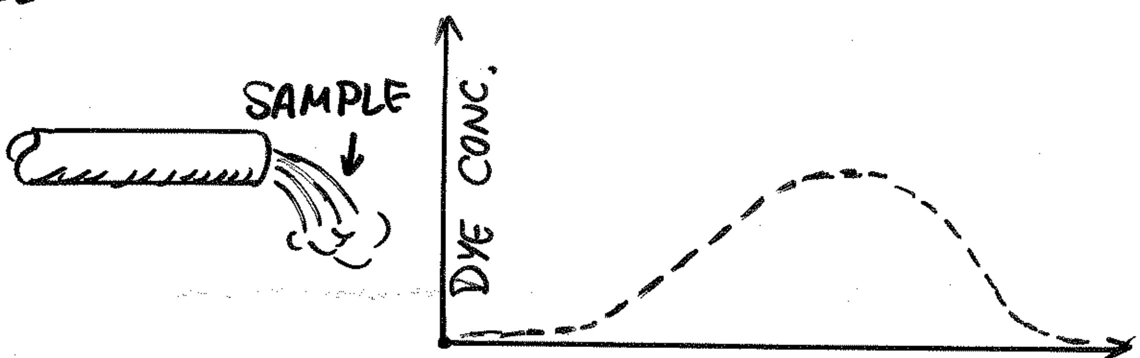
STAGE THREE

Dispersion: $\boxed{\text{Transport effected by variations in the mean velocity field}}$

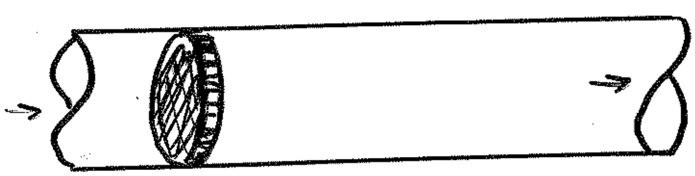
E.g., "Taylor Pipe Dispersion"

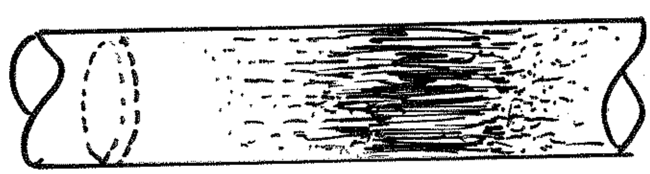






Results are the same "as if" diffusion (Fickian) were greatly enhanced.





Let's pretend... SIMPLE 1-D FICKIAN DIFFUSION

Lots easier to calculate than the complete, unsteady, 2D, actual dye patch

• TURBULENT DIFFUSION
• DISPERSION

Governing eqns. are <u>same</u> as for molecular diffusion, with <u>two main exceptions</u>:

1. Use empirical (much larger) values for diffusivity

$$D \longrightarrow K_e \longrightarrow E$$

MOLECULAR DIFFUSIVITY — EDDY DIFFUSIVITY — DISPERSIVITY

(ALL m^2/s)

2. Must modify SOLUTIONS of gov. eqns. to reflect reality of this substitution.

$D \approx$ CONSTANT, STEADY AND ISOTROPIC

$\left.\begin{array}{l} K_e \\ E \end{array}\right\}$ • SPATIALLY VARIABLE • MAY BE UNSTEADY • ANISOTROPIC