## Important Points

- Cartesian vector analysis is often used to solve problems in three dimensions.
- The positive directions of the $x$, $y$, $z$ axes are defined by the Cartesian unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.
- The *magnitude* of a Cartesian vector is $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$.
- The *direction* of a Cartesian vector is specified using coordinate direction angles $\alpha, \beta, \gamma$ which the tail of the vector makes with the positive $x$, $y$, $z$ axes, respectively. The components of the unit vector $\mathbf{u}_A = \mathbf{A}/A$ represent the direction cosines of $\alpha, \beta, \gamma$. Only two of the angles $\alpha, \beta, \gamma$ have to be specified. The third angle is determined from the relationship $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- Sometimes the direction of a vector is defined using the two angles $\theta$ and $\phi$ as in Fig. 2–28. In this case the vector components are obtained by vector resolution using trigonometry.
- To find the *resultant* of a concurrent force system, express each force as a Cartesian vector and add the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of all the forces in the system.

## EXAMPLE 2.8

Express the force **F** shown in Fig. 2–30 as a Cartesian vector.

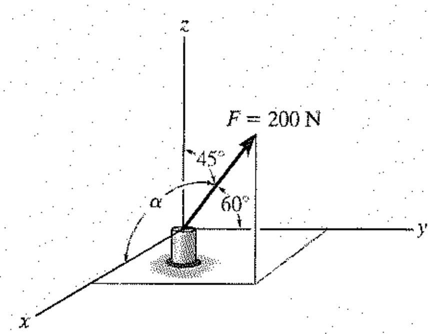



**Fig. 2–30**

SOLUTION

Since only two coordinate direction angles are specified, the third angle $\alpha$ must be determined from Eq. 2–8; i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60^\circ \quad \text{or} \quad \alpha = \cos^{-1}(-0.5) = 120^\circ$$

By inspection it is necessary that $\alpha = 60^\circ$, since $\mathbf{F}_x$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\begin{aligned}\mathbf{F} &= F\cos\alpha\,\mathbf{i} + F\cos\beta\,\mathbf{j} + F\cos\gamma\,\mathbf{k}\\ &= (200\cos 60^\circ\text{ N})\mathbf{i} + (200\cos 60^\circ\text{ N})\mathbf{j} + (200\cos 45^\circ\text{ N})\mathbf{k}\\ &= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}\text{ N} \qquad \textit{Ans.}\end{aligned}$$

Show that indeed the magnitude of $F = 200$ N.

$\sqrt{(100)^2 + (100)^2 + (141.4)^2} = 200$

2

## EXAMPLE 2.9

Determine the magnitude and the coordinate direction angles of the resultant force acting on the ring in Fig. 2–31*a*.

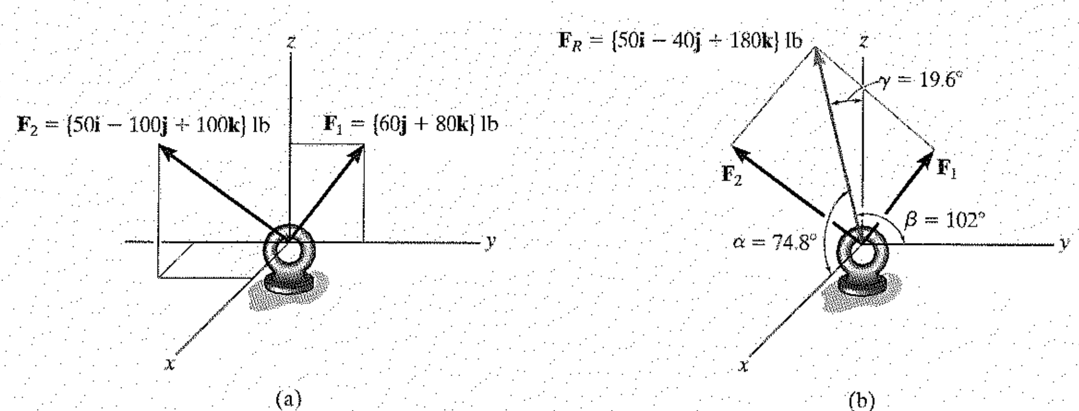



Fig. 2–31

### SOLUTION

Since each force is represented in Cartesian vector form, the resultant force, shown in Fig. 2–31*b*, is

$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{60\mathbf{j} + 80\mathbf{k}\}\text{ lb} + \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{ lb}$$
$$= \{50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}\}\text{ lb}$$

The magnitude of $\mathbf{F}_R$ is

$$F_R = \sqrt{(50\text{ lb})^2 + (-40\text{ lb})^2 + (180\text{ lb})^2} = 191.0\text{ lb}$$
$$= 191\text{ lb} \qquad \textit{Ans.}$$

The coordinate direction angles $\alpha, \beta, \gamma$ are determined from the components of the unit vector acting in the direction of $\mathbf{F}_R$.

$$\mathbf{u}_{F_R} = \frac{\mathbf{F}_R}{F_R} = \frac{50}{191.0}\mathbf{i} - \frac{40}{191.0}\mathbf{j} + \frac{180}{191.0}\mathbf{k}$$
$$= 0.2617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$$

so that

$$\cos\alpha = 0.2617 \qquad \alpha = 74.8° \qquad \textit{Ans.}$$
$$\cos\beta = -0.2094 \qquad \beta = 102° \qquad \textit{Ans.}$$
$$\cos\gamma = 0.9422 \qquad \gamma = 19.6° \qquad \textit{Ans.}$$

These angles are shown in Fig. 2–31*b*.

**NOTE:** In particular, notice that $\beta > 90°$ since the $\mathbf{j}$ component of $\mathbf{u}_{F_R}$ is negative. This seems reasonable considering how $\mathbf{F}_1$ and $\mathbf{F}_2$ add according to the parallelogram law.

## Important Points

- The dot product is used to determine the angle between two vectors or the projection of a vector in a specified direction.
- If vectors **A** and **B** are expressed in Cartesian vector form, the dot product is determined by multiplying the respective $x, y, z$ scalar components and algebraically adding the results, i.e., $\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z$.
- From the definition of the dot product, the angle formed between the tails of vectors **A** and **B** is $\theta = \cos^{-1}(\mathbf{A} \cdot \mathbf{B}/AB)$.
- The magnitude of the projection of vector **A** along a line $aa$ whose direction is specified by $\mathbf{u}_a$ is determined from the dot product $A_a = \mathbf{A} \cdot \mathbf{u}_a$.

## EXAMPLE 2.16

Determine the magnitudes of the projection of the force **F** in Fig. 2–44 onto the $u$ and $v$ axes.

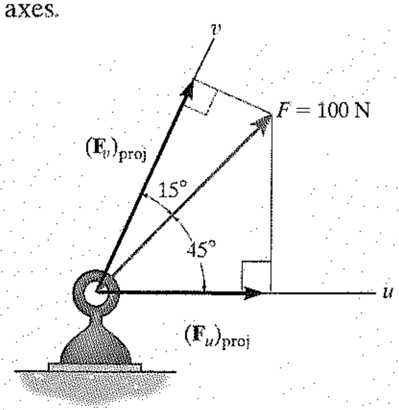



Fig. 2–44

### SOLUTION

**Projections of Force.** The graphical representation of the *projections* is shown in Fig. 2–44. From this figure, the magnitudes of the projections of **F** onto the $u$ and $v$ axes can be obtained by trigonometry:

$$(F_u)_{\text{proj}} = (100\text{ N})\cos 45^\circ = 70.7\text{ N} \qquad \textit{Ans.}$$

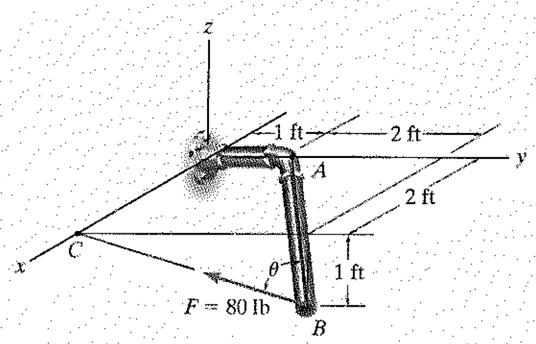
$$(F_v)_{\text{proj}} = (100\text{ N})\cos 15^\circ = 96.6\text{ N} \qquad \textit{Ans.}$$

**NOTE:** These projections are not equal to the magnitudes of the components of force **F** along the $u$ and $v$ axes found from the parallelogram law. They will only be equal if the $u$ and $v$ axes are *perpendicular* to one another.

NOT COMPONENTS OF F B/C u & V not ⊥

## EXAMPLE 2.18

The pipe in Fig. 2–46*a* is subjected to the force of $F = 80$ lb. Determine the angle $\theta$ between **F** and the pipe segment *BA* and the projection of **F** along this segment.




(a)

### SOLUTION

**Angle $\theta$.** First we will establish position vectors from *B* to *A* and *B* to *C*; Fig. 2–46*b*. Then we will determine the angle $\theta$ between the tails of these two vectors.

$(0-2)i + (1-3)j + (0-(-1))k$

$$\mathbf{r}_{BA} = \{-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}\}\ \text{ft},\ r_{BA} = 3\ \text{ft}$$

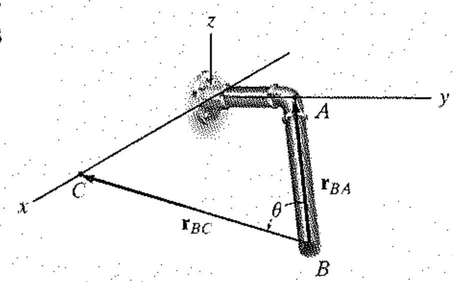
$$\mathbf{r}_{BC} = \{-3\mathbf{j} + 1\mathbf{k}\}\ \text{ft},\ r_{BC} = \sqrt{10}\ \text{ft}$$

$(2-2)i + (0-3)j + (0-1)k$

Thus,

$$\cos\theta = \frac{\mathbf{r}_{BA}\cdot\mathbf{r}_{BC}}{r_{BA}r_{BC}} = \frac{(-2)(0) + (-2)(-3) + (1)(1)}{3\sqrt{10}} = 0.7379$$

$$\theta = 42.5^\circ \qquad \textit{Ans.}$$




(b)

**Components of F.** The component of **F** along *BA* is shown in Fig. 2–46*c*. We must first formulate the unit vector along *BA* and force **F** as Cartesian vectors.

$$\mathbf{u}_{BA} = \frac{\mathbf{r}_{BA}}{r_{BA}} = \frac{(-2\mathbf{i} - 2\mathbf{j} + 1\mathbf{k})}{3} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

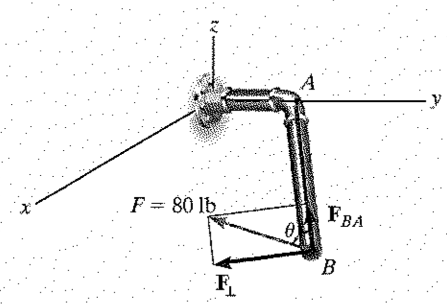
$$\mathbf{F} = 80\ \text{lb}\left(\frac{\mathbf{r}_{BC}}{r_{BC}}\right) = 80\left(\frac{-3\mathbf{j} + 1\mathbf{k}}{\sqrt{10}}\right) = -75.89\mathbf{j} + 25.30\mathbf{k}$$

Thus,

$$F_{BA} = \mathbf{F}\cdot\mathbf{u}_{BA} = (-75.89\mathbf{j} + 25.30\mathbf{k})\cdot\left(-\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$= 0\left(-\frac{2}{3}\right) + (-75.89)\left(-\frac{2}{3}\right) + (25.30)\left(\frac{1}{3}\right)$$

$$= 59.0\ \text{lb} \qquad \textit{Ans.}$$




(c)

**Fig. 2–46**

**NOTE:** Since $\theta$ has been calculated, then also, $F_{BA} = F\cos\theta = 80\ \text{lb}\cos 42.5^\circ = 59.0$ lb.