Vibrations

Topics

- Finding the natural frequency of a system
  - Force / Acceleration methods
  - Energy method
  - Equivalent mass-spring system

- Response of an un-damped system in free vibration

- Response of a damped system in free vibration
  - The system response when under-damped
  - The system response when critically damped
  - The system response when over-damped
  - Logarithmic Decrement

- Forced Response of Damped Systems
  - Forced Response of Damped Systems to Rotating Imbalance mass
  - Force Transmissibility

- Mass Response to Base Vibration

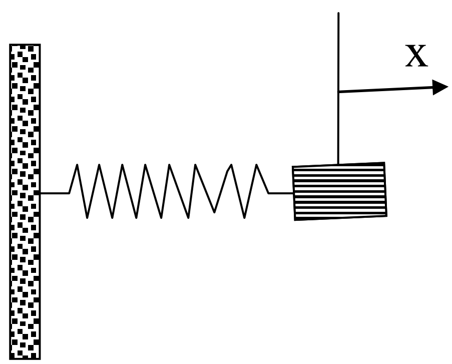
- Seismic Effect of base Vibration

Finding the natural frequency of a system

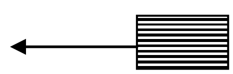
There are two common ways of finding the natural frequency of a single degree of freedom system: 1) based on the equation of motion and 2) using the energy method.

Using the Equation of Motion

For simple systems that do not involve multiple link and masses connected together, this method is simple to apply. The simplest system is shown below:



If we move the mass by a distance X to the right where X is measured from the free length of the spring, and release it, the FBD of the mass is



and the equation of motion is

$$-kx = m\ddot{x}$$

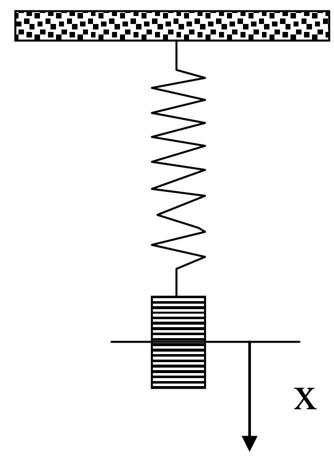
$$m\ddot{x} + kx = 0$$

Once the equation of motion is derived in this format, the natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m}}$$

For a hanging mass, we get the same natural frequency:

Measuring X from the static equilibrium



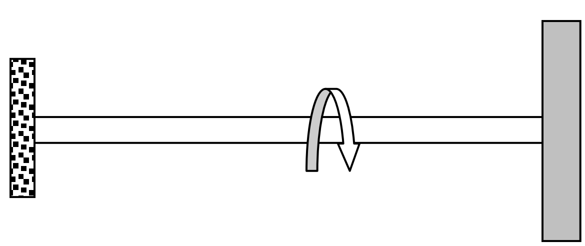
$$mg - k(x + x_0) = m\ddot{x}$$

$$Note\ that \quad kx_0 = mg$$

$$m\ddot{x} + kx = 0$$

In systems in which hanging masses are balanced by springs, the equations can be derived more quickly by "Ignoring the weight of the mass and assuming that at the spring is at free-length at the position of static equilibrium". This is a technique that also works with energy methods for deriving the equations of motions.

Rotational systems are treated similarly except for changing *m* with *I* and *x* with θ.



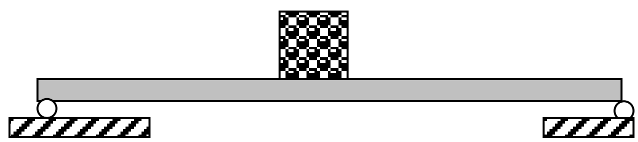
Show that the natural frequency of this disk and shaft is

$$\omega_n = \sqrt{\frac{k_t}{I}}$$

What are the units for k_t and I?

Problem #V1

Determine the natural frequency of oscillation of a 10-ft steel beam of standard 2" by 2" square cross-section with ¼ inch thickness when it supports a 100-lb weight in the its middle with the two ends free to rotate. Ignore the mass of the beam.



Answer: 54.17 rad/sec

Problem #V2

Determine the natural frequency of oscillation of the 10 foot, 100-lb bar connected to the end spring with k=500 lbs/ft and a torsional spring at the pivot point having a spring constant of 50 ft-lb per full turn.

Answer: 3.5 cycles/sec

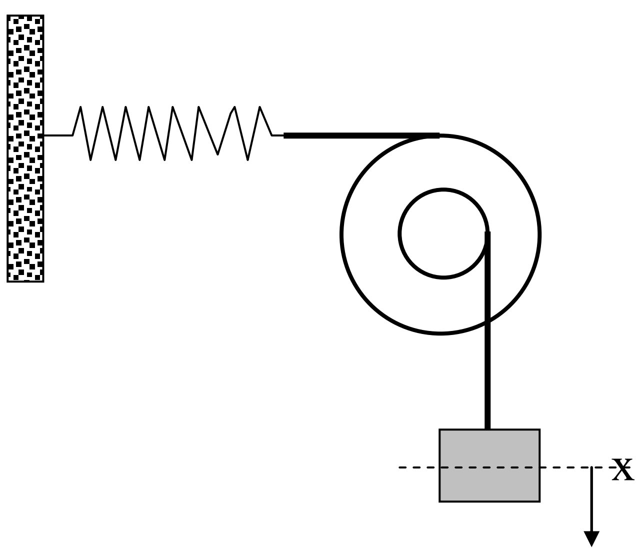
Problem #V3

Find the natural frequency of oscillations of a pendulum when it makes small angle oscillations. The pendulum weighs 1 lb and the length of the string is 2 feet, Ignore the mass of the string.

Answer:0.64 cycles/sec

Using the energy method for natural frequencies

When all the forces acting on a mass can not be easily determined, the derivation of the equation of motion using FBDs become cumbersome, slow, and error-prone. This is because to determine all the forces, the system has to be taken apart into multiple pieces and each FBD must be solved to determine the forces necessary to set up the equation of motion. For example, consider the following case:

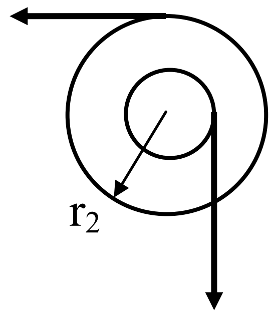


The block mass is M, the disk's mass moment of inertia is I and the smaller and larger radii are r_1 and r_2 and the spring constant is K.

To determine the natural frequency, we can use the FBD of the block and derive the following:

$$-T = m\ddot{x}$$

Note that the weight of the block is ignored by pretending that the spring is at free length. This is based on the technique underlined before. Now we need the FBD of the disk



$$-kr_2\theta + Tr_1 = I\ddot{\theta}$$

From the kinematics of unwinding a string from a pulley, we know that:

$$\ddot{x} = r_1\ddot{\theta}$$

Substituting in the first equation we get:

$$\left(mr_1^2 + I\right)\ddot{\theta} + kr_2^2\theta = 0$$

The natural frequency then becomes

$$\omega_n = \sqrt{\frac{kr_2^2}{(mr_1^2 + I)}}$$

Note that we had to draw two FBDs, we had to make sure the direction of forces and accelerations are selected consistently, and then solve the resulting equations to find the natural frequency. The energy method removes this long error-prone procedure and only requires a kinematic relationship between the system elements to find its kinetic energy. The energy method would be illustrated next using the same example. We have the choice of developing the equation of motion based on θ or based on x. This solution uses θ. Assume the system has a $\theta, \dot{\theta}, and\ \ddot{\theta}$

The kinetic energy of the system is:

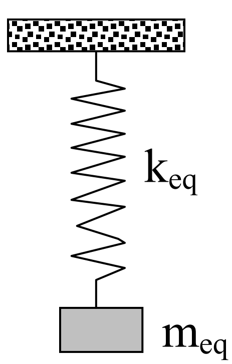
$$T = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m(r_1\dot{\theta})^2 \quad = I_{eq}\dot{\theta}^2$$

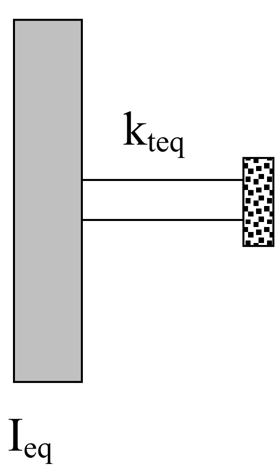
Similarly the potential energy of the system is:

$$U = \frac{1}{2} k(r_2\theta)^2 \quad = k_{teq}\theta^2$$

Note that for any hanging mass system, the gravitational energy is ignored and the spring is assumed to be at free length in static equilibrium.

We can use the kinetic and potential energies of the system to find equivalent masses (for translational DOF) or inertias (for rotational DOF). We can also use the potential energy to find the equivalent linear springs (for translational DOF) or torsional springs (for rotational DOF).





The equivalent inertia is:

$$I_{eq} = (mr_1^2 + I)$$

The equivalent torsional spring is:

$$PE = \frac{1}{2}k(r_2\theta)^2 \quad and \quad K_{teq} = kr_2^2$$

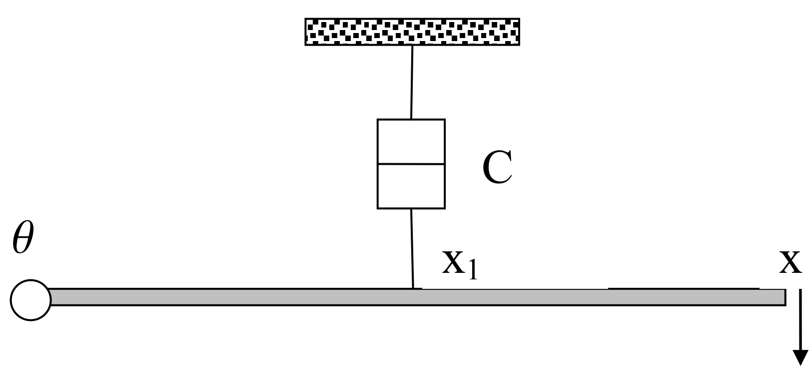
This equivalent system yields the following natural freq.

$$\omega_n = \sqrt{\frac{kr_2^2}{(mr_1^2 + I)}}$$

Equivalent Viscous Damping

$$DE = \frac{1}{2}C_{teq}\dot{\theta}^2$$

$$DE = \frac{1}{2}C_{eq}\dot{x}^2$$



$$DE = \frac{1}{2}C\dot{x}_1^2 = \frac{1}{2}C\left(\frac{L}{2}\right)^2\dot{\theta}^2 = \frac{1}{2}\left(\frac{CL^2}{4}\right)\dot{\theta}^2 = \frac{1}{2}C_{teq}\dot{\theta}^2$$

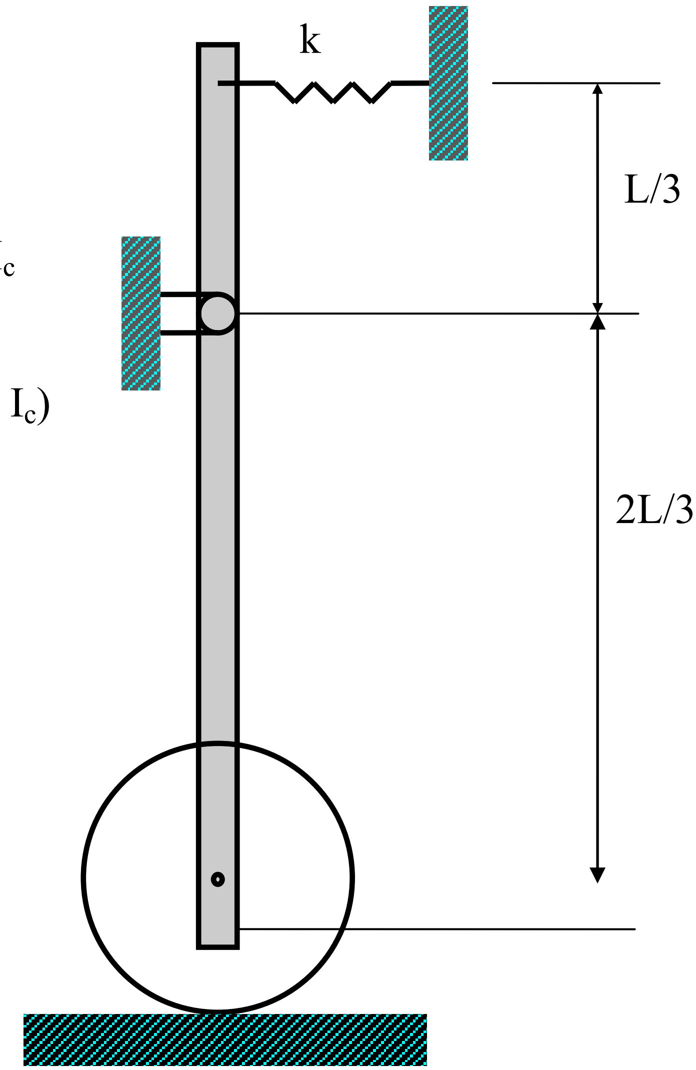
$$DE = \frac{1}{2}C\dot{x}_1^2 = \frac{1}{2}C\left(\frac{1}{2}\right)^2\dot{x}^2 = \frac{1}{2}C_{eq}\dot{x}^2 \quad \Rightarrow C_{eq} = \frac{C}{4}$$

Problem #V4

Find the natural frequency of this system in terms of the following variables.

Rod's mass = m
Rod's I w/r to pivot = I_R
Cylinder mass = M
Cylinder's I w/r to its CG= I_c
Radius of Cylinder = R

(Do not substitute for I_R and I_c)

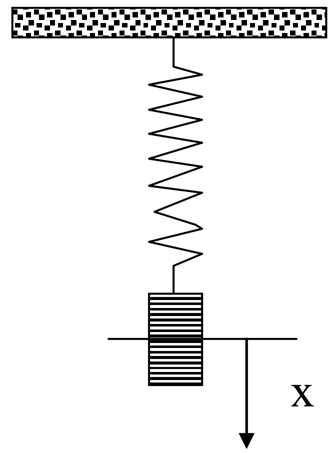


Answer: $I_{eq} = \frac{2KE}{\dot{\theta}^2} = I_R + \frac{4}{9}ML^2 + \frac{4}{9}I_c L^2 R^2$

$$k_{teq} = \frac{1}{9}kL^2$$

Response of an un-damped system in free vibration

Equation of motions:



$$\ddot{x} + \frac{k}{m} x = 0$$

The response is:

$$x(t) = A_0 \sin(\omega_n t + \phi_n)$$

where

$$A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2}$$

Where x_0 is the initial displacement and $\dot{x}_0$ is the initial velocity of the mass. The measurement for *x* is from the static equilibrium of the mass.

Rotational system

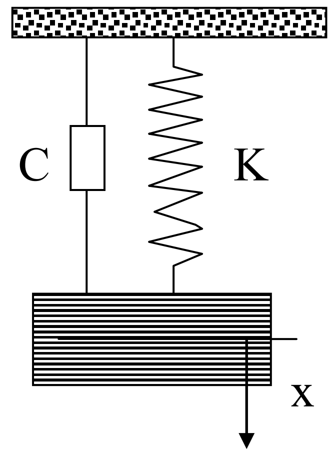
$$\ddot{\theta} + \frac{k_t}{I}\theta = 0 \qquad where \quad \omega_n = \sqrt{\frac{k_t}{I}}$$

Where the angles are measured in radians.

Response of a damped system in free vibration

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = 0$$



The natural frequency ω_n is the same as the un-damped system.

c is the *damping coefficient* in the units of lbs per in/sec. The unit-less *damping ratio* is:

$$\xi = \frac{c}{2m\omega_n}$$

The *damped natural frequency* is:

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n$$

The system response when under-damped: $\xi < 1$

$$x(t) = e^{-\xi\omega_n t}(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d}\sin\omega_d t)$$

The system response when critically damped: $\xi = 1$

$$x(t) = e^{-\omega_n t}(x_0 + (\dot{x}_0 + \omega_n x_0)t)$$

The system response when over-damped: $\xi > 1$

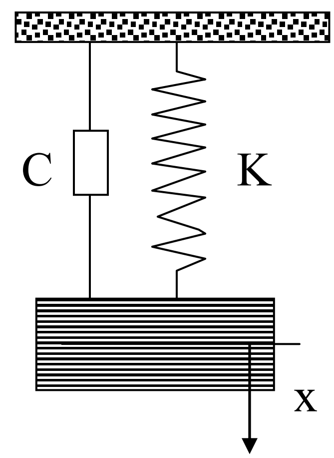
$$x(t) = C_1 e^{(-\xi + \sqrt{\xi^2 - 1})\omega_n t} + C_2 e^{(-\xi - \sqrt{\xi^2 - 1})\omega_n t}$$

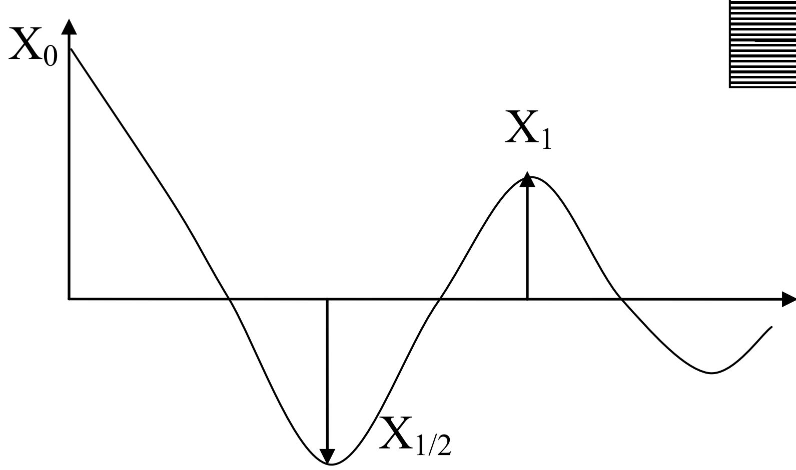
C_1 and C_2 are the constants that are lengthy in closed-form. They can be found numerically by the initial conditions.

Logarithmic Decrement

Logarithmic decrement is a unit-less characterizer of an under-damped system. Logarithmic decrement is:

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$





The logarithmic decrement relates the peak amplitudes (n=1,1.5,2, etc) as the mass oscillates according to:

$$\frac{X_n}{X_0} = e^{-n\delta}$$

Problem # V5

The figure shows a model for a gate. It is composed of a rod with a concentrated end mass, a torsional spring, and a torsional damper. The parameters of the system are:

m = Mass of the uniform rod = 10 kg [0.6854 slugs]
M = concentrated mass = 2 kg [0.137 slugs]
k = 20 N-m/rad. [14.752 ft-lbs/rad]

Determine:

a) Natural frequency of the system

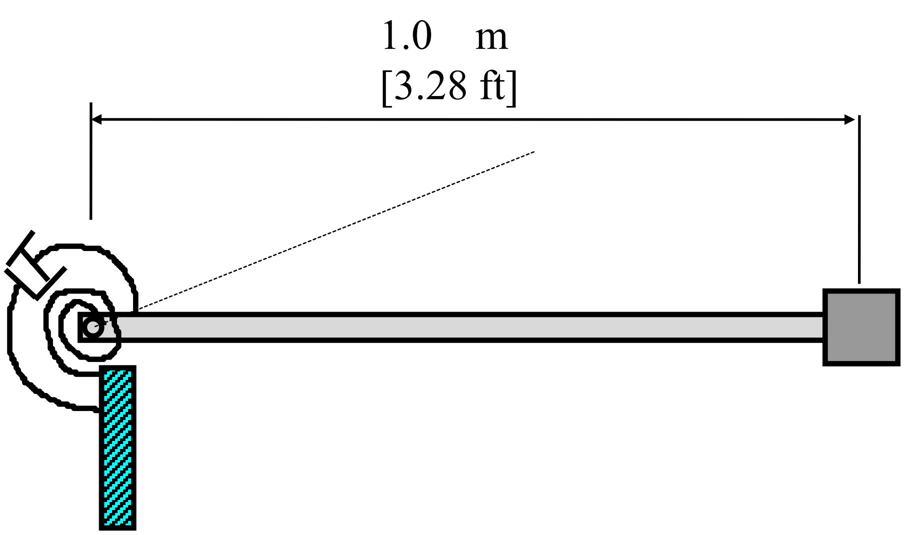
If the door is opened 75 degrees and released without any initial velocity.

For a critically damped system determine:

b) The angle of rotation w/r to closed position after 2 seconds.

For an under-damped system with a damping ratio of 0.5 determine:

c) Damped natural frequency of the system
d) The 1st overshoot (in degrees) relative to the closed position using logarithmic decrement relationship.



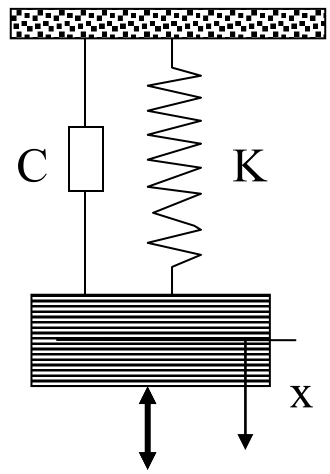
Answers: a) 1.937 rad/sec, b) 7.6 deg,
c) 1.68 rad/sec, d) 12.2 deg.

Forced Response of Damped Systems

The mass oscillates with same frequency as the forcing function.

There is a phase shift between the mass and the forcing function.

The mass amplitude peaks near the Forcing frequency. The smaller the damping ratio, the larger the amplitude.



$F=F_0\text{Sin}(\omega t)$

$$x(t) = X\sin(\omega t + \phi)$$

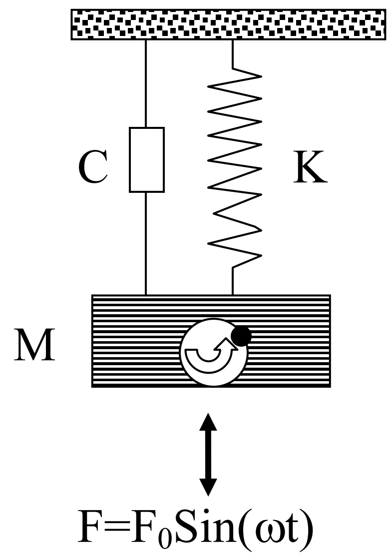
The amplitude is:

$$X = \frac{F_0 / K}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

where $r = \omega / \omega_n$

Plot the amplitude versus the frequency ratio using MathCad for different values of the damping ratio.

Forced Response of Damped Systems to Rotating Imbalance mass



Motors are not perfectly balanced and create inertially-induced sinusoidal forcing functions.

$$F = F_0 \sin(\omega t)$$

$$F_0 = me\omega^2$$

The magnitude of this forcing function is the centrifugal force as indicated. In this relation *me* is the mass times eccentricity (mass imbalance) and is usually given in *lb-in* units which should be converted to slugs-in units.

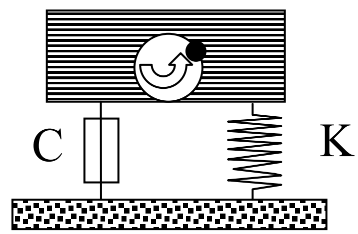
The system response is

$$X = \frac{\frac{em}{M} r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

Force Transmissibility

Any machine with rotating masses transmits forces to its foundation. The more this transmitted force, the more shaking is felt through the floor which can be annoying or adversely affect other precision machines.

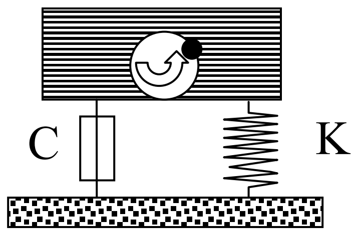
The force transmitted to the ground is the summation of the spring force and the force through dashpot. Force Transmissibility is the ratio of the transmitted force to the induced force:



$$T_r = \frac{F_T}{F_0} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

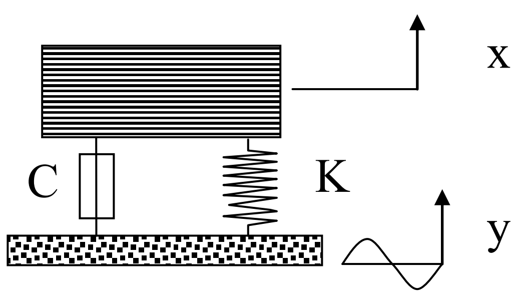
Problem #V6

The motor of a 100 lb machine is running at 1000 rpm. The motor has a load imbalance of 20 lb-in. The damping ratio of the supporting material is 0.20 and would be a constant. Determine K, the spring constant for the foundation that would reduce the transmitted force to the ground by 90%. Also, determine the amplitude of the transmitted force.



Answers: 128.2 lbs/in , 57.1 lbs

Mass Response to Base Vibration



A harmonic base vibration creates a harmonic system (mass) vibrations. Given the amplitude of the based motion and its frequency, we can find the amplitude of the mass – its frequency of motion is the same as the base motion. The mass motion amplitude is:

$$\frac{X}{Y} = \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

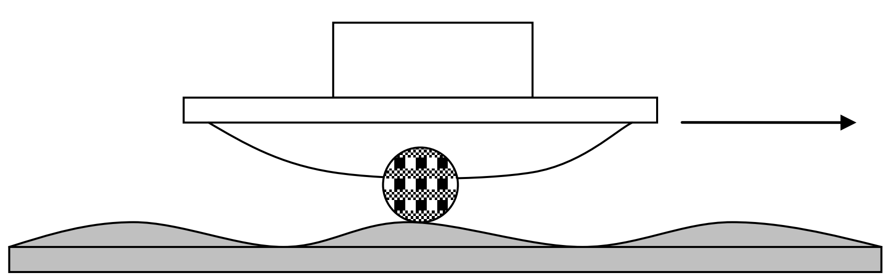
Where X is the amplitude of the mass and Y is the amplitude of the base motion, and *r* is the frequency ratio

Force transmitted to the base in base vibration is:

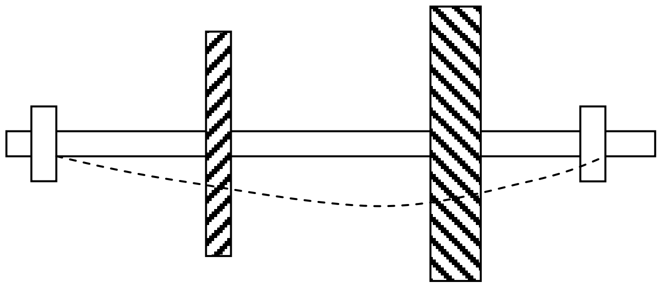
$$T_r = \frac{F_T}{kY} = r^2 \frac{\sqrt{1+(2\xi r)^2}}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

Problem #V7

A trailer is being pulled by a car traveling at 55 mph. The road contour has a sine wave profile with an amplitude (peak to peak) of 0.5" and a period (distance between two peaks) of 10 feet. The damping ratio of the trailer suspension is 0.050 and it has a natural frequency of 16 rad/sec including the load it carries. Determine the amplitude of vibration of the trailer and the load.
Ans: 0.06 inches



Natural Frequency of Rotating Shafts



When a rotating shaft carries a number of weights, the first natural frequency of the shaft's lateral vibration can be found by the following approximate formula:

$$f = \frac{1}{2\pi}\sqrt{\frac{g(W_1\delta_1 + W_2\delta_2 + \ldots + W_n\delta_n}{W_1\delta_1^2 + W_2\delta_2^2 + \ldots + W_n\delta_n^2}}$$

Where *f* is the frequency in Hz, *g* is the gravitational acceleration, Ws are the weights of the hanging element, and δs are the static deflection under the weights only (exclude the deflection due to all external forces).

Problem #V8

Determine the first natural frequency of the two pulley system shown above when the shaft is made of 2" diameter steel, W_1=80 lbs, W_2=120 lbs, the distance between W_1 and left bearing support is 30", the distance between the two weights is 40", and the distance between W_2 and the right bearing support is 20 inches. Answer: f=11.8 cycles/sec

The beat phenomenon

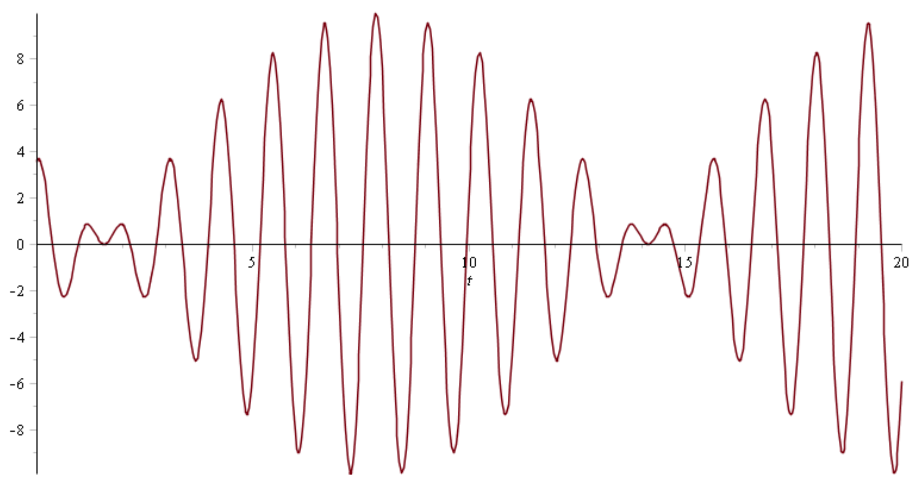
If two machines working at slightly different frequencies are next to each other, their transmitted vibration to the ground would intensify and subside due to the beat phenomenon. Beat happens because the two amplitudes add and subtract regularly creating the beat effect. The beat frequency is the difference between the two constituent frequencies.

The following plot shows the response of two sinusoidal waves with a frequency difference of 0.5 radians per second. The beat frequency is 0.5 rad/sec or 0.08 Hz. The period is calculated to be 12.56 seconds. Similar situation happens with sound waves.

$Y := 5 \cdot \sin(5\,t) + 5 \cdot \cos\left(5.5 \cdot t + \frac{\mathrm{Pi}}{4}\right);$

$$5 \sin(5\,t) + 5 \cos\left(5.5\,t + \frac{1}{4}\pi\right)$$

$plot(Y, t = 0\,..20);$



$FrequencyRadians := \mathrm{abs}(omega1 - omega2);$

$$|\omega1 - \omega2|$$

$omega1 := 5;\ omega2 := 5.5;$

$$5$$
$$5.5$$

$FrequencyRadians;$

$$0.5$$

$FrequencyHertz := evalf\left(\frac{FrequencyRadians}{2 \cdot \mathrm{Pi}}\right);$

$$0.07957747152$$

$Period := \frac{1}{FrequencyHertz};$

$$12.56637062$$