PE Review Notes – 2013

Particle, Particle System, and Rigid Body

The equations of dynamics fall into two categories:

- *single particle formulations*,
- *particle system formulations*.

Single particle formulation of motion is simpler than the particle system formulation and if the problem can be adequately solved using single particle formulation, that formulation would lead to faster and easier solution.

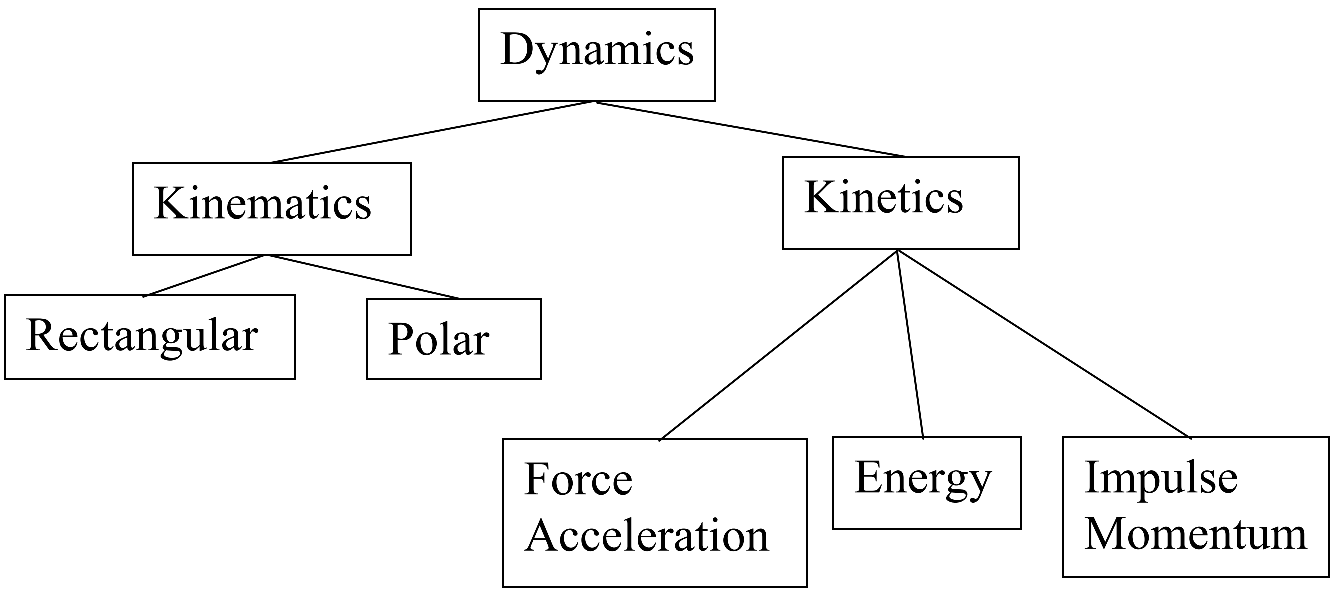
Therefore, in dynamics study, we do not physically define the distinction between a particle and a particle system. We only need to make a distinction between:

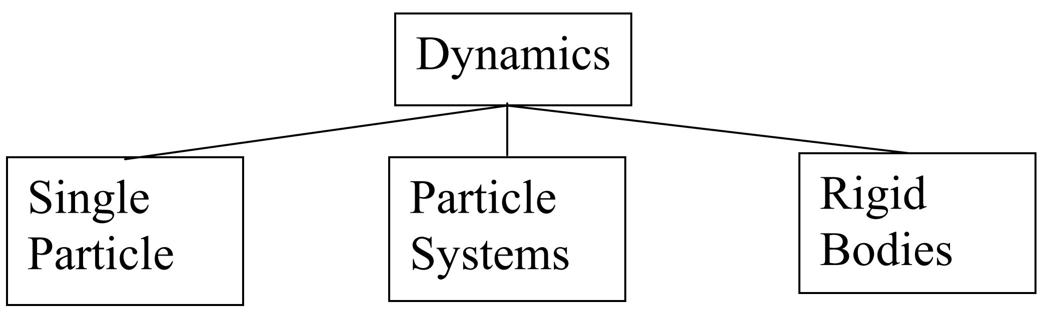
- *Particle-problems*
- *Particle-system problems.*

Different questions about the same object can lead to different applicable formulations. For example, the questions involving the motion of a car travelling on a road can often be solved using single particle formulation. Question involving the behaviour of the same car motion in a rollover situation in a side impact requires particle system formulation to be used.

With a little experience and reading the rest of these notes, you would be able to easily recognize the best formulations to use.

Clasification of Dynamics Problems

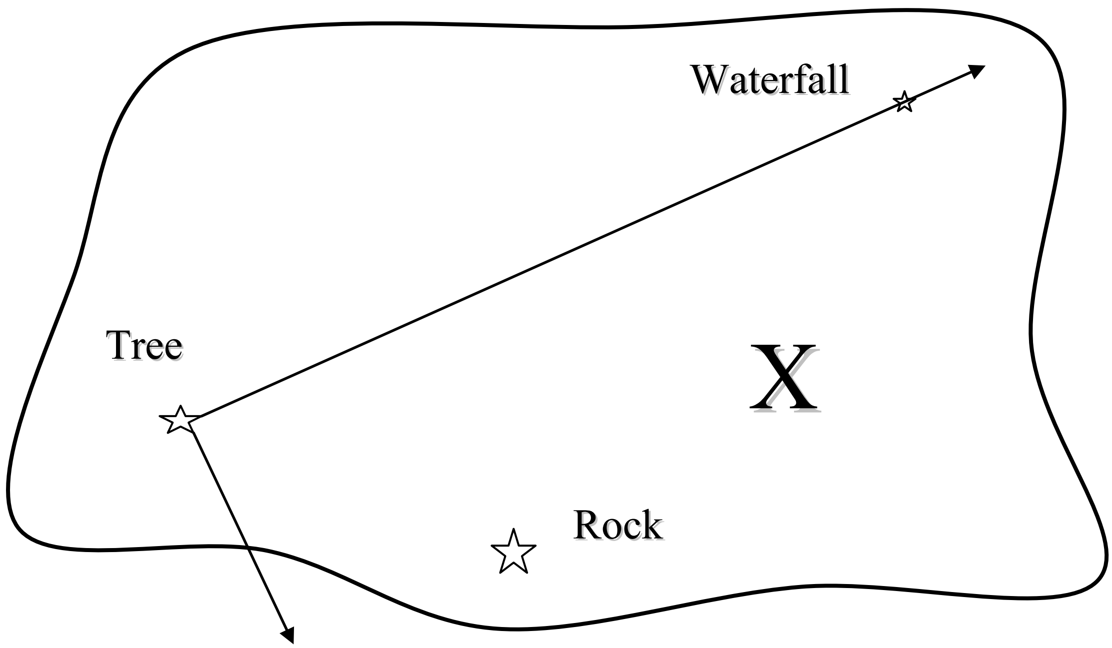




$\Sigma \mathbf{F}=m\mathbf{a}$ $\Delta U=\Delta KE+\Delta PE$ $\Sigma \mathbf{F}\Delta t = m\Delta \mathbf{V}$	$\Sigma \mathbf{F}=m\mathbf{a_G}$ $\Delta U=\Delta KE+\Delta PE$ $\mathbf{F_i}\Delta t = m_i\Delta \mathbf{V_i}$ $\mathbf{F_i R_i}\Delta t = m_i\mathbf{R_i}\Delta \mathbf{V_i}$	$\Sigma \mathbf{F}=m\mathbf{a_G}$ $\Delta U=\Delta KE+\Delta PE$ $\mathbf{F}\Delta t = m\Delta \mathbf{V_G}$ $\mathbf{M} = I\alpha$

Coordinate Systems

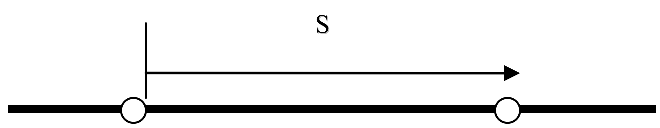
A coordinate system is a frame of reference. It needs an origin marker, a marker for x-axis identification, and a marker for positive Y-axis identification. Imagine you are on an island and thinking of burying a treasure you discovered.



You can use a big tree as the origin. For the x-axis you can use a waterfall far away as the marker. The y-axis is normal to the x-axis but what defines the positive direction needs another marker and this can be a big rock or mountain. Normally we need a marker for positive z-axis too but in this case it is obvious which direction the treasure is burried. You can now disclose the location of the treasure in three standard methods – we only look at Cartesian and cylindrical coordiante methods.

Analyzing Particle Motion

1D Rectilinear Motion



Constant Linear Acceleration Motion

$$V = V_0 + at$$

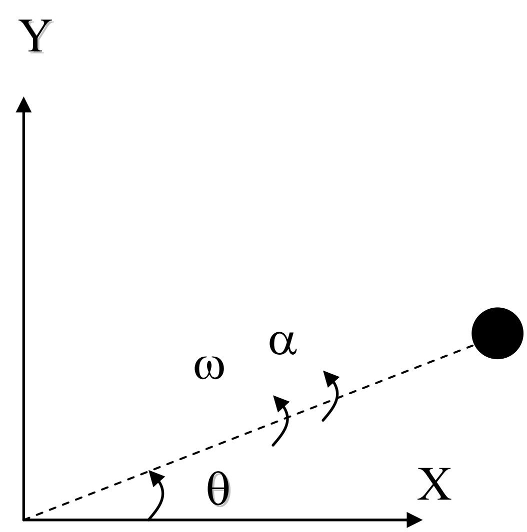
$$S = S_0 + V_0 t + \frac{1}{2} a t^2$$

$$V^2 = V_0^2 + 2aS$$

1D Angular Motion

$$\omega = \dot{\theta} = \frac{d\theta}{dt}$$

$$\alpha = \ddot{\theta} = \frac{d\omega}{dt}$$



Constant Angular Acceleration Motion

$$\omega = \omega_0 + \alpha t$$

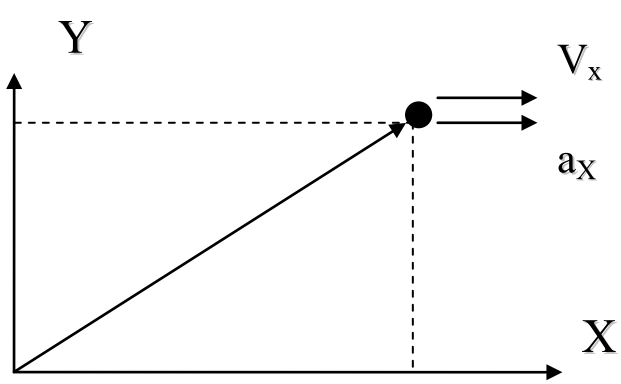
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

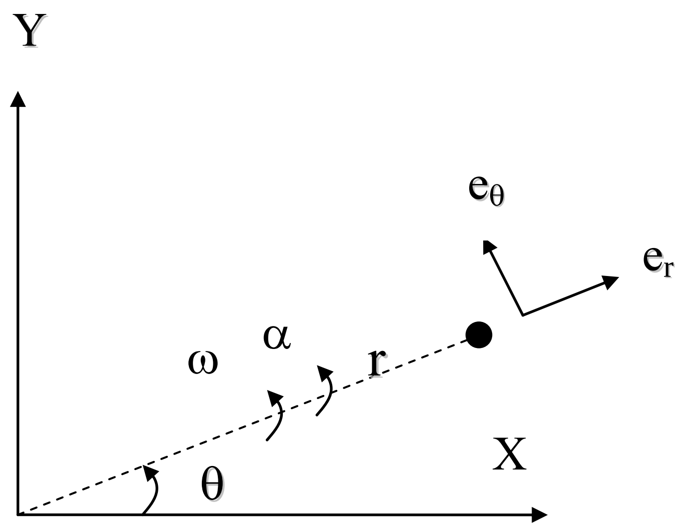
2D and 3D Motion in Cartesian (x/y/z) Coordiates

$$V_x = \dot{x} \qquad a_x = \ddot{x}$$

$$V_y = \dot{y} \qquad a_y = \ddot{y}$$



2D and 3D Motion in Polar and Cylindrical Coordinates



Velocity and acceleration components are given in the direction of unit vectors along radial and transverse (tangential) directions.

$$V_r = \dot{r} \qquad a_r = \ddot{r} - r\dot{\theta}^2$$

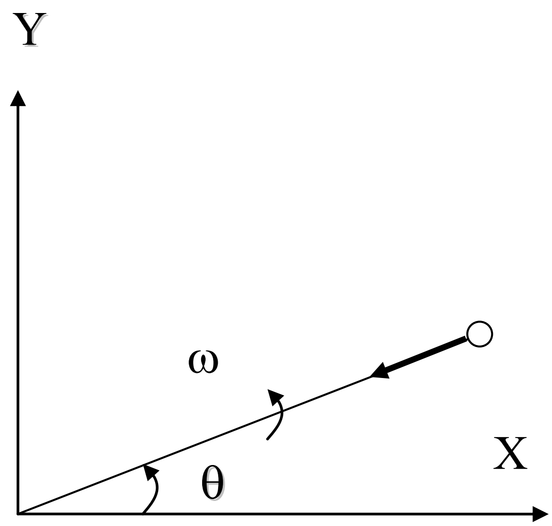
$$V_\theta = r\dot{\theta} \qquad a_\theta = r\alpha + 2\dot{r}\dot{\theta}$$

When the path of motion is circular, the radial velocity and accelerations become zero.

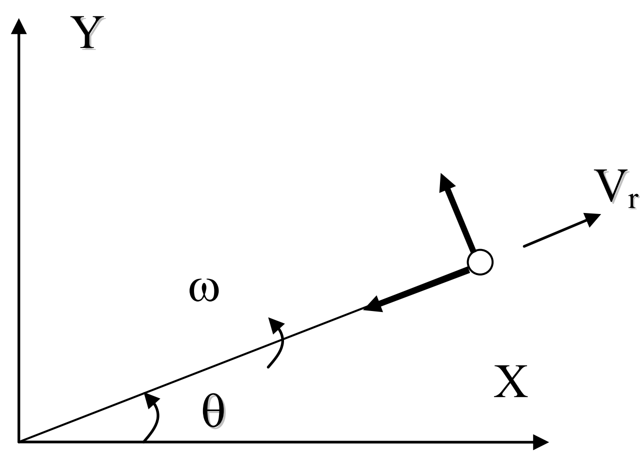
$$V_r = 0 \qquad a_r = -r\dot{\theta}^2 = -\frac{V^2}{r}$$

$$V_\theta = V = r\dot{\theta} \qquad a_\theta = r\alpha$$

If an object rotate at zero angular acceleration (α=0) in a circular path, its acceleration is the usual centripital component – a vector pointing toward the center.



If an object moves out radially at constant speed on a uniformly CCW rotating disk, its acceleration would be the usual coriolis acceleration pointing to its left.

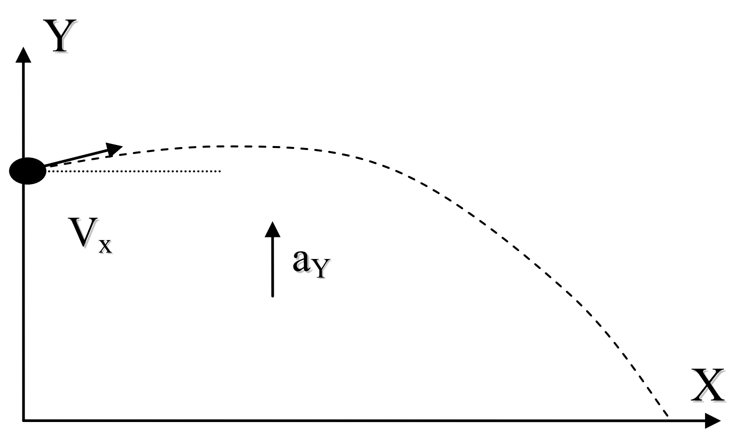


Problem #D1 : 2D motion that is best solved in Cartesian formulation of motion

Projectile Motion

D#1. A cannon is fired at a 30 degree angle with horizon from a height of 200 feet (60.96 m) at a muzzle velocity of 400 ft/s (121.92 m/s). Find the time to impact the ground and the distance travelled neglecting the air resistance.

Hints: Motion in X-direction (horizontal) is a constant acceleration motion with an acceleration of zero. Motion in Y-direction is also a constant acceleration motion with an acceleration of *g* downward.



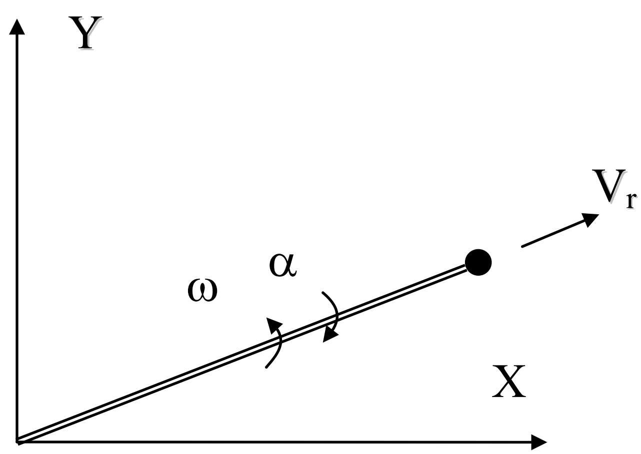
Answers: time=13.3 sec , Distance=4626 feet (1410 m)

D#2. A car starts from rest on a horizontal circular road with an acceleration of 7 ft/s^2 (2.1336 m/s^2). The road radius is 300 ft (91.44 m). How long would it take for the car to reach

an acceleration of magnitude g/4? What is the velocity?
Answer: 4.93 sec at V=34.5 ft/s (10.5156 m/s)

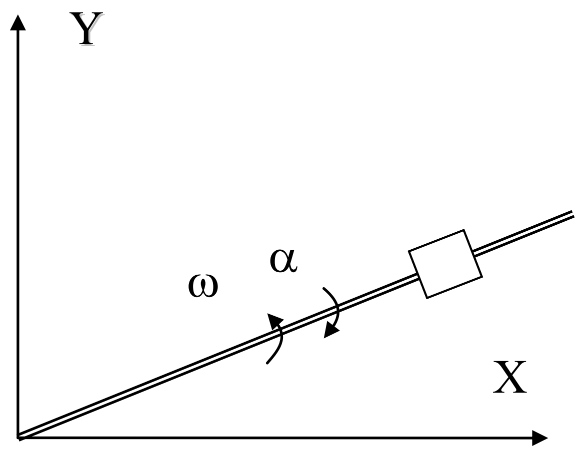
Hints: Tangential acceleration is given and radial acceleration is a function of velocity. Velocity is a function of time and tangential acceleration.

D#3. At an instant a rod of 9 inches (0.2286 m) length is rotating at 10 rad/s CCW and slowing down at a rate of 40 rad/s^2. Find the speed of the end of the rod – speed is the magnitude of the velocity vector. Also find the magnitude of the acceleration of the end of the rod.

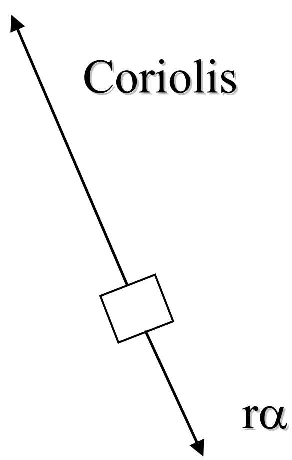


Answers: V=7.5 ft/s (2.286 m/s) and a=80.8 ft/s^2 (24.62784 m/s^2)

D#4. At the instance shown, a slider is 18 inches (0.4572 m) from the pivot point and is sliding outward at a velocity of 15 ft/s (4.572 m/s) on a rod while speeding up at a rate of 180 ft/s^2 (54.864 m/s2). The rod is rotating CCW at 10 rad/s and slowing down at the rate of 40 rad/s^2. Find the magnitude of the acceleration vector of the slider.

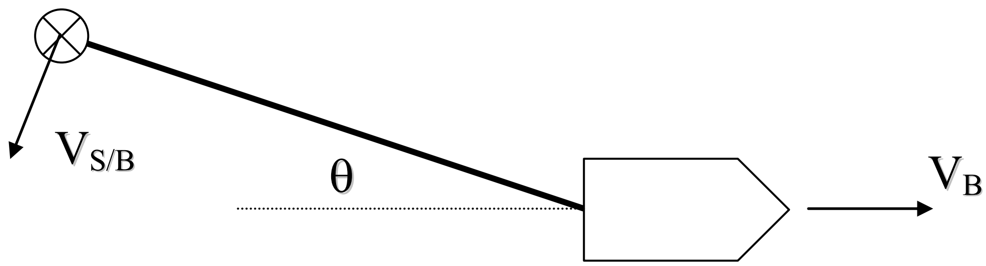


Answer: 242 ft/s^2 (73.7616 m/s2)



Relative Motion Formulation

In many problems a complex motion can be expressed in terms of two or more simpler motions. A typical situation is the motion of a water skier relative to the ground coordiante system as shwn below:



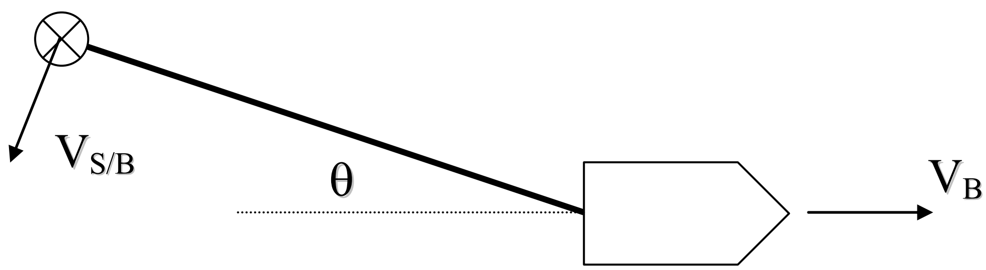
The motion of the skier is simple with respect to the boat (it goes through a circular path relative to the boat). And, the motion of the boat is simple relative to the ground. Therefore, we can express the motion of the skier with respect to the groud using the relative velocity and acceleration relationships and two simple vectors:

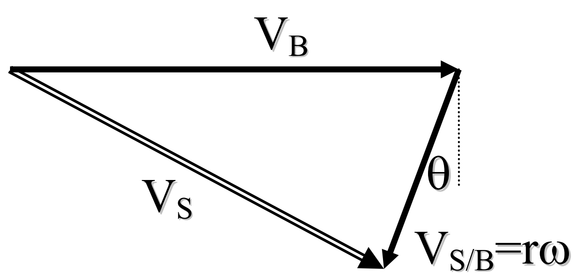
$$\vec{V}_S = \vec{V}_B + \vec{V}_{S/B}$$

$$\vec{a}_S = \vec{a}_B + \vec{a}_{S/B}$$

The easiest and quickest method of solving simple vector relationships is through graphical means.

Example: Find the skier's speed and magnitude of acceleration if the speed of the boat is V_B and it is accelerating at the rate of a_B. The skier's angular velocity is ω and is a constant. The rope length is r.

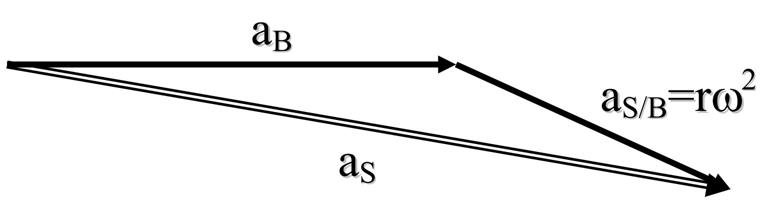




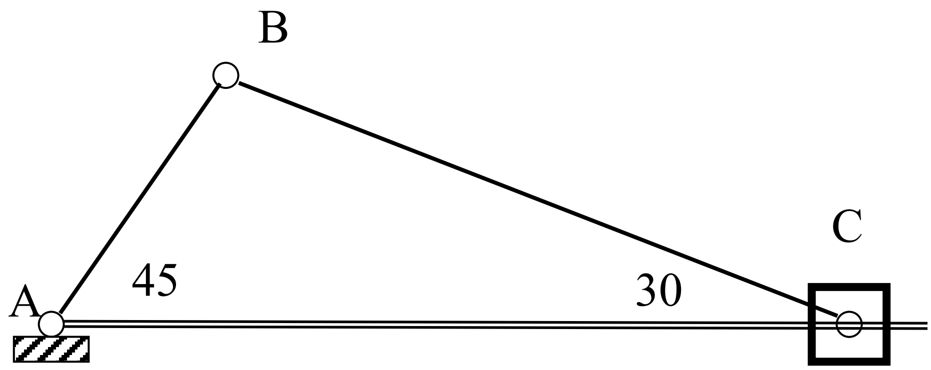
The skier's square of speed is:

$$V^2 = V_B^2 + (r\omega)^2 - 2V_B r\omega \cos(90 - \theta)$$

The acceleration of the skier is determined from the following vector diagram:



Example: In the slider crank mechanism shown, the crank is 18 inches (0.4572 m) and at an angle of 45 degrees. It is rotating in CCW direction at a constant speed of 100 rad/s. The connecting rod is at an angle of 30 degrees with the horizontal line. Find the speed of the slider C and the angular velocity of the connecting rod at this instance.



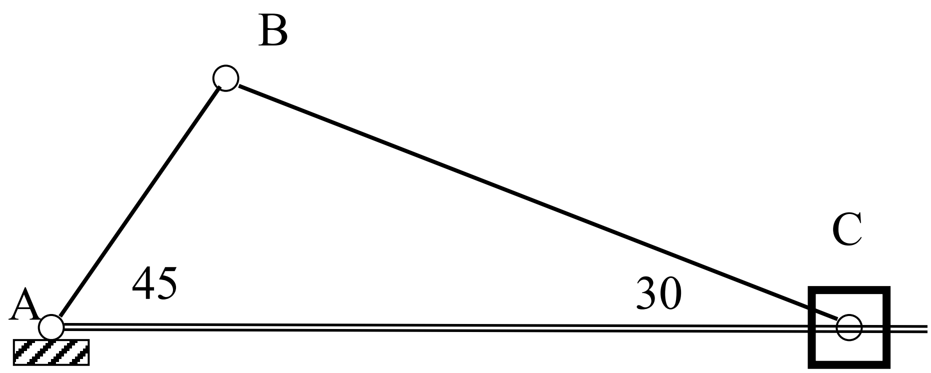
$$\left|V_{B/A}\right| = \left|V_B\right| = r\omega = (\frac{18}{12})(100) = 150\,ft/s\ (45.72m/s)$$

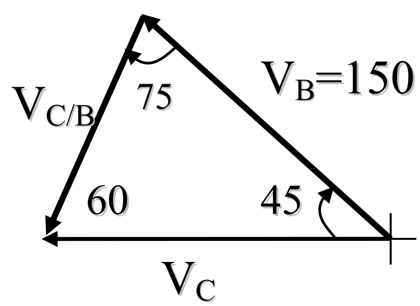
$$\vec{V}_C = \vec{V}_B + \vec{V}_{C/B}$$

A 2D vector equation can be solved if it has two unknow scalars (legnths or directions). This vector equation has two unknows:

Magnitude of V_C
Magnitude of $V_{C/B}$

A graphical method can be used to easily find both values.

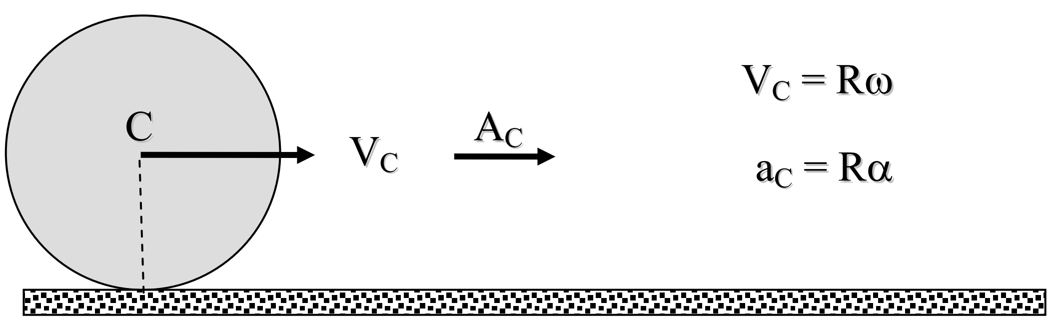




Using the law of sines

$$\frac{V_C}{Sin(75)} = \frac{150}{Sin(60)} \Rightarrow V_C = 167 \quad ft/s \quad 50.90\ m/s$$

Kinmatics of rolling motion (or rope and pully)



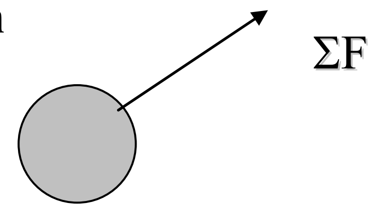
$V_C = R\omega$

$a_C = R\alpha$

Single Particle Kinetics

Force – Acceleration Formulation

$$\sum \vec{F} = m\vec{a}$$



F is the force required to bring about the acceleration ***a***. The acceleration is measured in a fixed (non-accelerating) frame of reference. This is called the equation of motion.

In Cartesian formulation of acceleration:

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

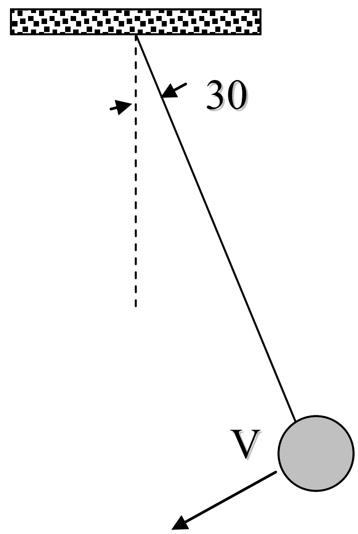
In Polar formulation of acceleration:

$$\sum F_r = ma_r$$

$$\sum F_\theta = ma_\theta$$

This formulation of kinetics is used when the instantaneous relationship between the force and acceleration is required.

D#5: A 25 lb (111.2 N) ball is hanging from a 2 ft (0.61 m) rope. At the instant shown the speed of the ball is 13 ft/s (3.96 m/s). Find the rope tension (T) and angular acceleration of rope.



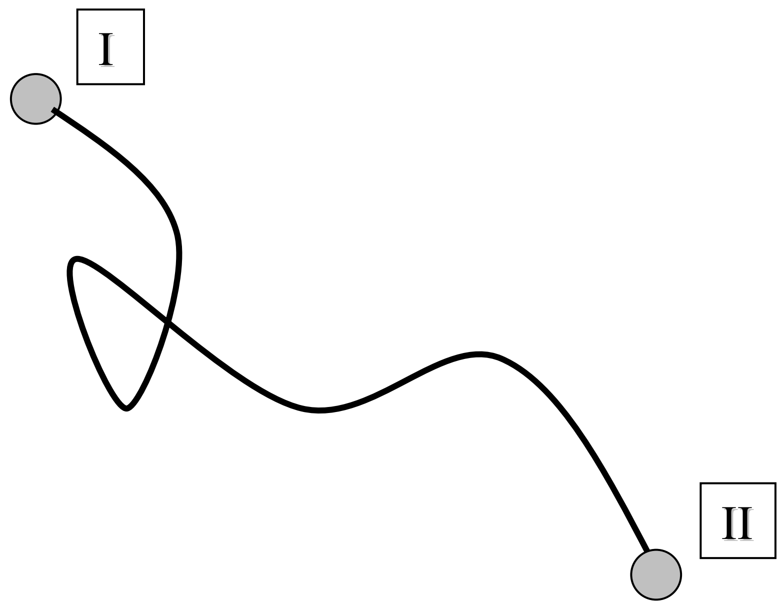
Hint: Draw the Free Body Diagram (FBD) of the ball. From the velocity information find radial acceleration. Write the equation of motion along the direction of the rope to find T. Write the equation of motion along the path of the particle (θ direction) and find the angular acceleration. Answer: 87.25 lbs (388.1 N) and 8.05 rad/s^2

Note on Units

Force	**Mass**	**Distance**	**g_c**
Lbs	**Slugs (lb-s^2/ft)**	**ft**	**32.2 ft/s^2**
Lbs	**“Blobs”(lb-s^2/in)**	**in**	**386 in/s^2**
N	**Kg**	**meters**	**9.81 m/s^2**

Note: Unfortunately it is also customary to express force using the units of Kg. When Kg is used to indicate force, multiply it by 9.8 to convert it to Newtons. For example if your weight is 60 kg, your weight is about 600 Newtons.

Work/Energy Formulation of particle Kinetics



Work/Energy formulation of kinetics is obtained when the equation of motion (Force/Acceleration formulation) is integrated over the particle path from an initial position to a final position.

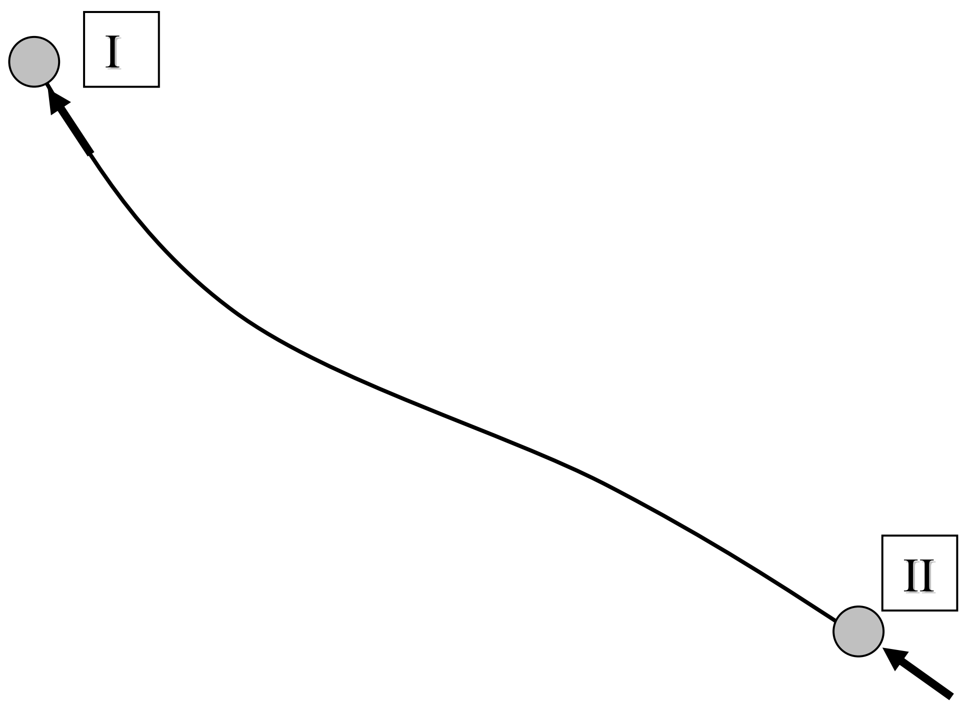
$$\int \sum \vec{F} d\vec{x} = \int m\vec{a} d\vec{x}$$

The left side of this equation is the work done on the particle during travel including the work by the gravitational force. The right hand side is the change in the kinetic energy of the paticle. The final form is:

$$U_{1-2} = \Delta KE + \Delta PE$$

Where:

$U_{1\text{-}2}$: Work done by all forces (except gravitational and spring forces).



For *friction force*, the work is:

$$U_{1\text{-}2}=-f\,d$$

Where ***f*** is the fiction force and ***d*** is the length of the path traveled. For a constant force *F* acting in the direction of travel

$$U_{1\text{-}2}= Fd$$

Energy and Power Conversions

- 1 ft-lb = 1.35 Jouls
- 1 ft-lb/sec = 1.35 Watts
- 1 hp = 746 Watts
- 1 hp-sec = 746 Jouls
- 1 BTU = 1055 Jouls
- 1 BTU/sec = 1055 Watts
- 1 Calorie = 4.19 Jouls
- 1 Calorie/sec = 4.19 Watts
- 1 Amp-hour@V volt = V Watt-hr =3600V Jouls

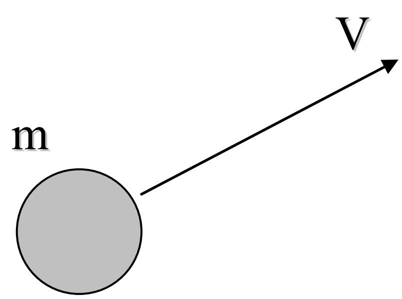
Example: Suppose you weigh 175 lbs. You decide to take the staris from the 1st floor to the 5th floor of EB (about 100 feet) . How much energy it takes in kilo-calories, jouls, Watt-hours, BTU, hp-sec, and Amp-hours @ 1.5 volts. Assume all the energy spent goes into elevation gain.

E_1 = 175(100) = 17500 ft-lb
E_2 = (17500)*(1.35) = 23625 Jouls (Watt-sec)
E_3 = 23625/4190 = 5.64 kilo-calories (called calories)
E_4 = 23625/3600 = 6.56 Watt-hr
E_5 = 23625/1055 = 22.4 BTU
E_6 = 23625/746 = 31.7 hp-sec
E_6 = 6.56/1.5 = 4.37 Ah @ 1.5 volts (2 AA size Alkalines)

ΔKE: Change in kinetic energy of the particle

The change in kinetic energy of a particle is:

$$\Delta KE = \frac{1}{2} m V_2^2 - \frac{1}{2} m V_1^2$$

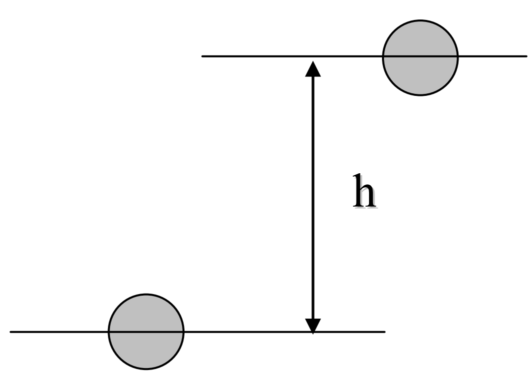


ΔPE: Change in potential energy of the gravitational and spring forces

Gravitational potential energy

$$\Delta PE_{gravity} = mg(h_2 - h_1)$$

When a particle moves up in the gravitational field, the gravitational force gains potential energy and ΔPE is postive.

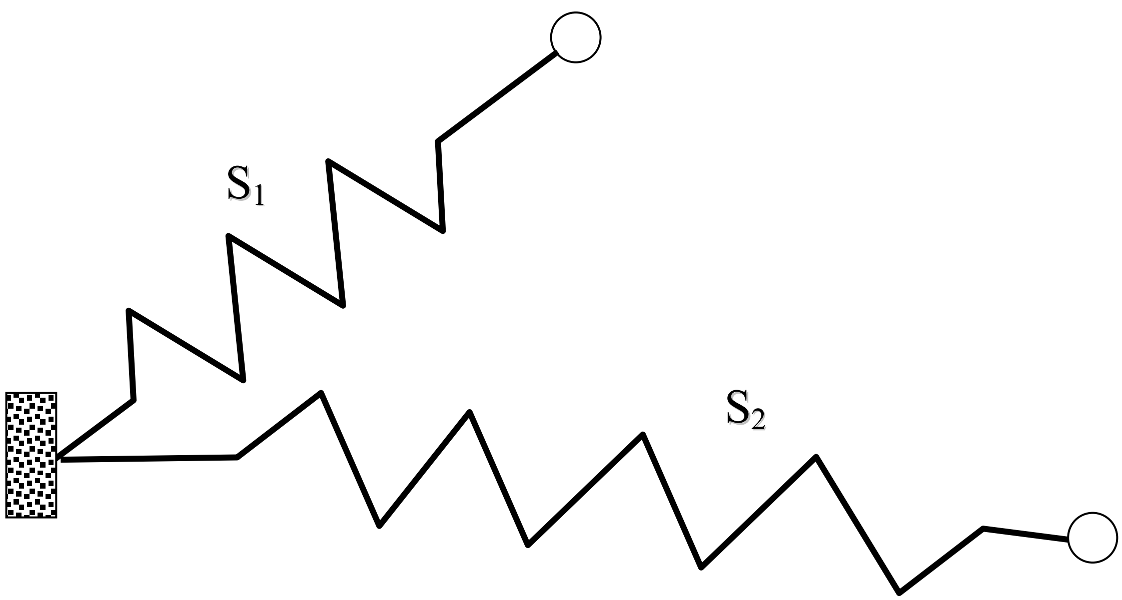


Elastic potential energy

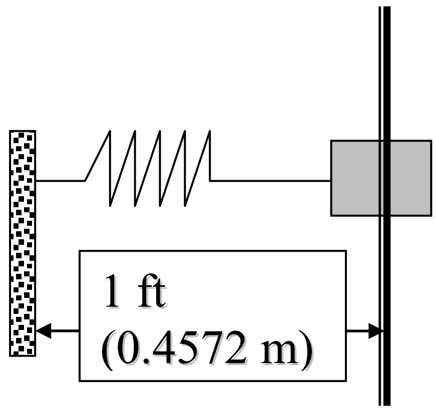
$$\Delta PE_{Elastic} = \frac{1}{2}k(\Delta S_2^2 - \Delta S_1^2)$$

ΔS_1 is the spring stretch – the difference in spring length at position-1 relative to the free length of the spring (S_0):

$\Delta S_1 = S_1 - S_0$
$\Delta S_2 = S_2 - S_0$

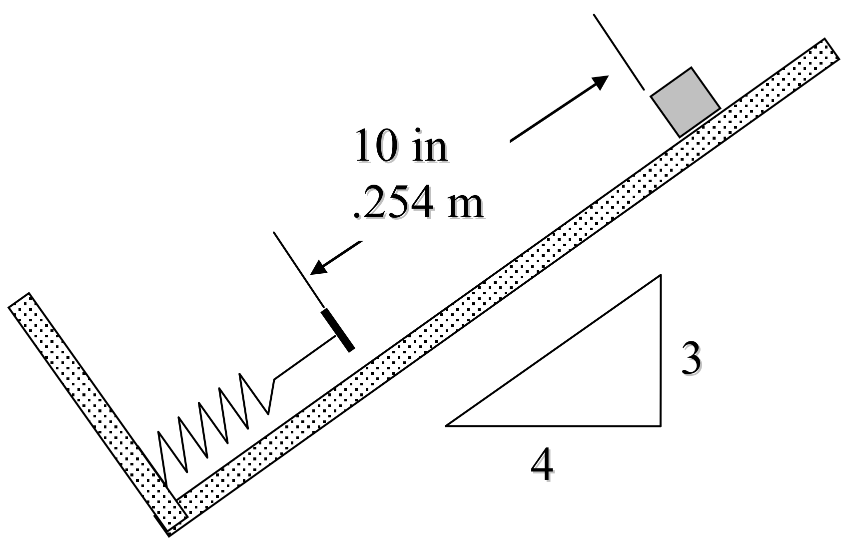


D#6: The spring shown is at free length at the position drawn and has a spring constant of 7 lbs/ft (102 N/m). The slider weighs 5 lbs (22.24 N). If the slider is released from rest, what would be its velocity after dropping 1.5ft (0.4572 m).



Answer: 8.2 ft/s (2.5 m/s)

#D7 A 25-lb (111.2 N) box is released with an intial velocity of 30 in/s (9.144 m/s) from the position shown. The spring constant is 100 lb/in (17500 N/m) and the incline has a slope of 3:4. The coefficient of dynamic friction between the slider and the path is 0.3. Find the maximum force in the spring after the block is released.



Answer: 163.7 lbs (728.1 N)

Impulse/Momentum Formulation of particle Kinetics

Impulse-Momemtum formulation of kinetics is obtained when the equation of motion (Force/Acceleration formulation) is integrated over some time period from t_1 to t_2.

$$\int \sum \vec{F} dt = \int m\vec{a} dt$$

The left side of this equation is the impluse of the external forces acting on a particle during the time period of interest. The right hand side is the change in the linear momentum of the paticle.

Assuming a constant mass, the RHS of the formula (the momentum) only depends on initial and final velocities:

$$\int \sum \vec{F} dt = m(\vec{V}_2 - \vec{V}_1)$$

Note that both the force and the velocities are vector quantities. These relationships can be writtten in terms of their components in (usually) Cartesian coordiates. Here is an example for one direction:

$$\int \sum \vec{F}_x dt = m(\vec{V}_{x2} - \vec{V}_{x1})$$

Also note that when a particle is not acted upon by any forces (in some direction), the particle momentum is conserved in that direction.

Example: A 2000 lb (8896 N) car is travelling at 60 mph (96.6 km/h) when the driver slams on the brakes resulting in all the wheels to lock. If the coefficient of friction between the tires and the road is 0.80 and assuming equal friction forces from all tires, how long would it take for the car to come to a stop?

Classic solution: The friction force is μN = 0.8(2000) = 1600 lbs. From the Newton's formula

$$F = ma$$

$$1600 = \left(\frac{2000}{32.2}\right)a \quad \Rightarrow \quad a = 25.76 \quad ft/s^2$$

The velocity is 88 ft/sec and it drops to zero. The time it takes is:

$$V = V_0 + at$$

$$88 = 0 + 25.76t \quad \Rightarrow \quad t = 3.4\,\sec$$

Alternatively using impulse/momentum

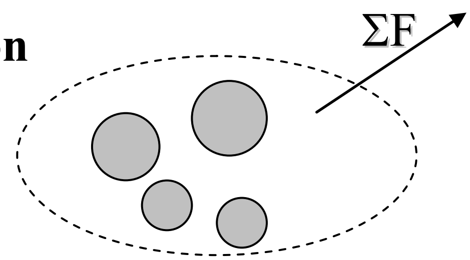
$$F\Delta t = m\Delta V$$

$$1600\Delta t = (\frac{2000}{32.2})(88) \quad \Rightarrow \quad \Delta t = 3.4 \quad \sec$$

Multiple Particle Kinetics

When the problem solution involves multiple particles, multiple-particle formulation of motion formulas would result in quicker and easier solutions. If the particle system is rigidly connected together, the formulas for the special case of rigid-bodies work even better.

Force – Acceleration Formulation



$$\sum \vec{F}_{ext} = m_t \vec{a}_G$$

ΣF is the summation of external forces required to bring about the acceleration $\boldsymbol{a}_G$ to the center of mass of the particle system of total mass m_t.

The problems that this Force-Acceleration addresses are similar to those of single particles.

Work/Energy Formulation of particle Kinetics

The energy formulation for the particle systems is also similar to single particles:

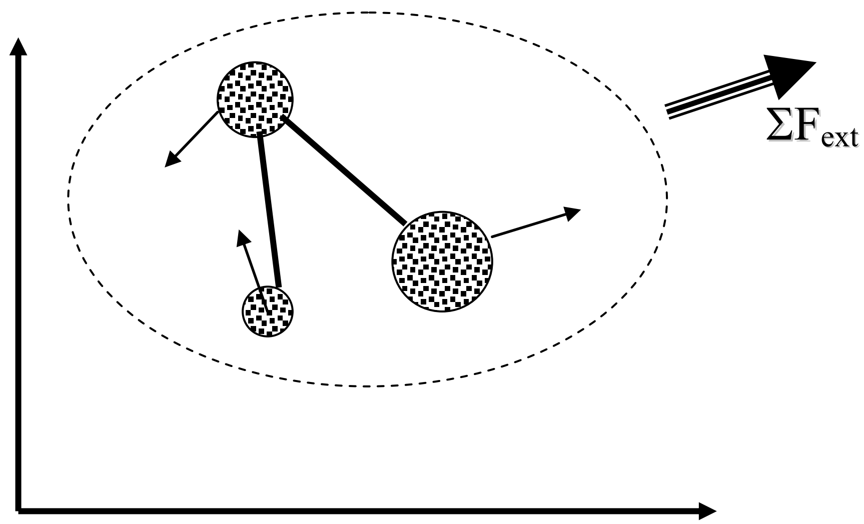
$$\sum U_i = \Sigma \Delta KE + \Sigma \Delta PE$$

The KE and the PE terms include the all the particles. The work includes the work done by external and internal forces on the particles . The work of internal forces, in general, may not cancel out - so **<u>apply only when frictional losses do not exist and no energy is created or lost within the system.</u>**

Impulse/Momentum Formulation of particle systems

Impulse-Momemtum formulation of kinetics also remains the same form:

$$\int \sum \vec{F}_{ext} dt = \sum m_i (\vec{V}_{i2} - \vec{V}_{i1})$$



The momentum term includes all particles. **<u>Fortunately the impulse term only includes the external forces as the impulse of internal forces cancel out to zero</u>**. The main application involves interaction of two or more particles such as in an impact and also in applications involving fluid flow in nozzles, jet propulsion, and rocket thrusts.

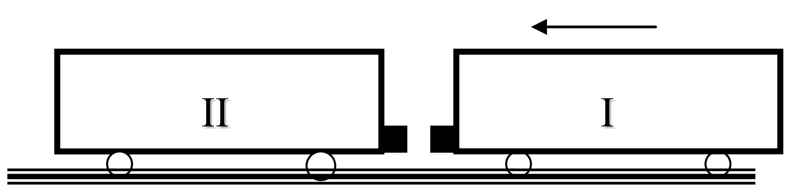
Note that both the force and the velocities are vector quantities. These relationships can be writtten in terms of their components usually in a Cartesian coordiates. Here is an example for one direction:

$$\int \sum \vec{F}_{x,ext} dt = \sum m_i (\vec{V}_{i,x2} - \vec{V}_{i,x1})$$

Conservation of Linear momentum

Any particle system that is not acted upon by any external forces in a certain direction would preserve its momentum in that direction.

Example: A railroad car travelling at 2 ft/s (0.6100 m/s) bumps another car of the same weight and the two cars latch together. What is the speeds of the two cars immediately after the impact?



$$\Delta l = (2m_1 V_{after}) - (m_1 V_1) = 0$$

$$V_{after} = \frac{1}{2} V_1 = 1\, ft/s \quad [0.30\, m/s]$$

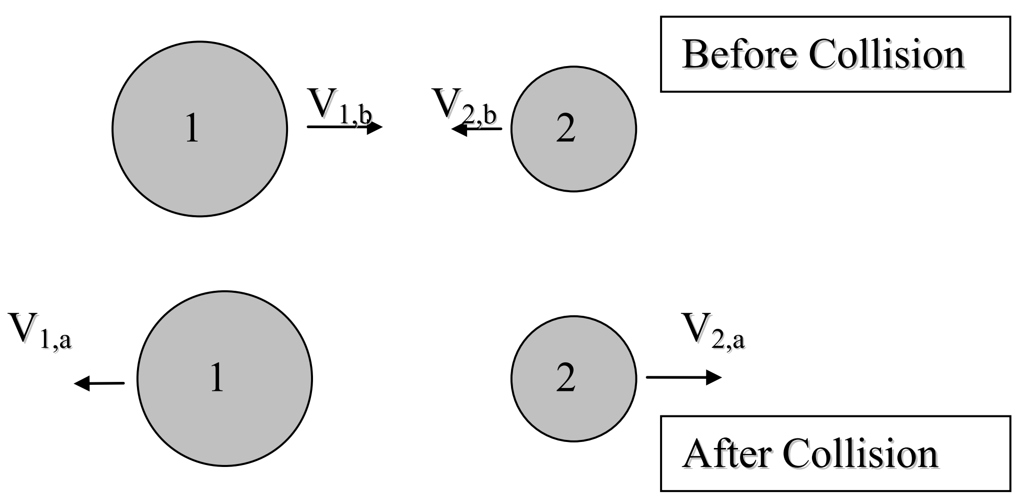
If the two cars do not latch together, they would not have the same speed after impact. The change in linear momentum becomes:

$$\Delta l = (m_1 V_{1,after} + m_2 V_{2,after}) - (m_1 V_1 + 0) = 0$$

This is one equation in two unknows. To solve this impact problem we need to know an experimental coefficient known as the *coefficient of restitution*. Coefficient of restitution measures the degree of impact elasticity.

Coefficient of restitution

Suppose two particles collide head on as shown below:

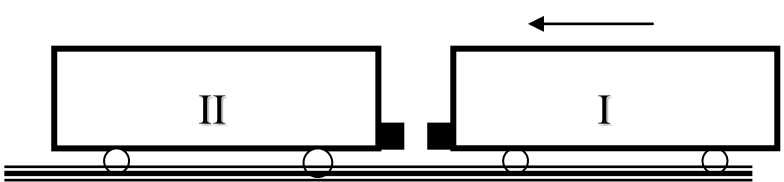


If the impact is **perfectly elastic**, minimal energy is lost and the relative velocity with which the paricles depart is the same as the relative velocity of the two particles approaching before impact.

If the impact is **perfectly plastic**, maximum energy is lost due to impact and the parts stick together. The relative velocity of departure becomes zero. In general, the coefficient of restitution is defined as:

$$e = \frac{|\mathrm{Re}\,lative\_speed\_of\_departure|}{|\mathrm{Re}\,lative\_speed\_of\_approach|}$$

Example: Consider the same problem of railroad cars. Assuming a coefficient of restitution of 0.8, What would be the speeds of the two cars immediately after the impact?



$$\Delta l = (m_1 V_{1,after} + m_2 V_{2,after}) - (m_1 V_{1,b} + 0) = 0$$

$$V_{1,a} + V_{2,a} = 2 \quad ft/s \qquad [0.61 \quad m/s]$$

The coefficient of restitution relationship is:

$$e = 0.8 = \frac{V_{2,a} - V_{1,a}}{V_{1,b}} \quad \Rightarrow \quad V_{2,a} - V_{1,a} = 1.6$$

Note: $V_{2,a}$ and V_{2b} are unknowns and they are assumed to be positive or in the same direction. From the two equations

$V_{2,a}$=1.8 ft/s [.55 m/s]

$V_{1,a}$=0.2 ft/s [.061 m/s]

Our assumption of both being positive and in the same direction was right. The bumping car loses almost all of its speed where the bumped car jumps ahead almost as fast as the approaching car before impact. Those who have played pool should be well familiar with this kind of impact.

Kinetics of Fluid Flow

Fluids are also particle systems and the impulse-momentum formulation applies to the mass within a control volume. In this review, we only consider systems in which the mass within control volume remains a constant. This includes fluid flow through nozzles but not rockets. For fluid flow through nozzles, the basic relationship is:

$$\vec{R} = \dot{m}(\vec{V}_{out} - \vec{V}_{in})$$

where **R** is the total resultant force that would bring about a change in fluid momentum within the control volume. The mass flow rate is $\dot{m}$ and is measured in slugs per second [or kg/s] and it can be calculated as:

$$\dot{m} = A\rho V$$

Where A is the flow (pipe) area, ρ is the mass density of the fluid, and V is the flow velocity. The last useful relationship is the conservation of mass:

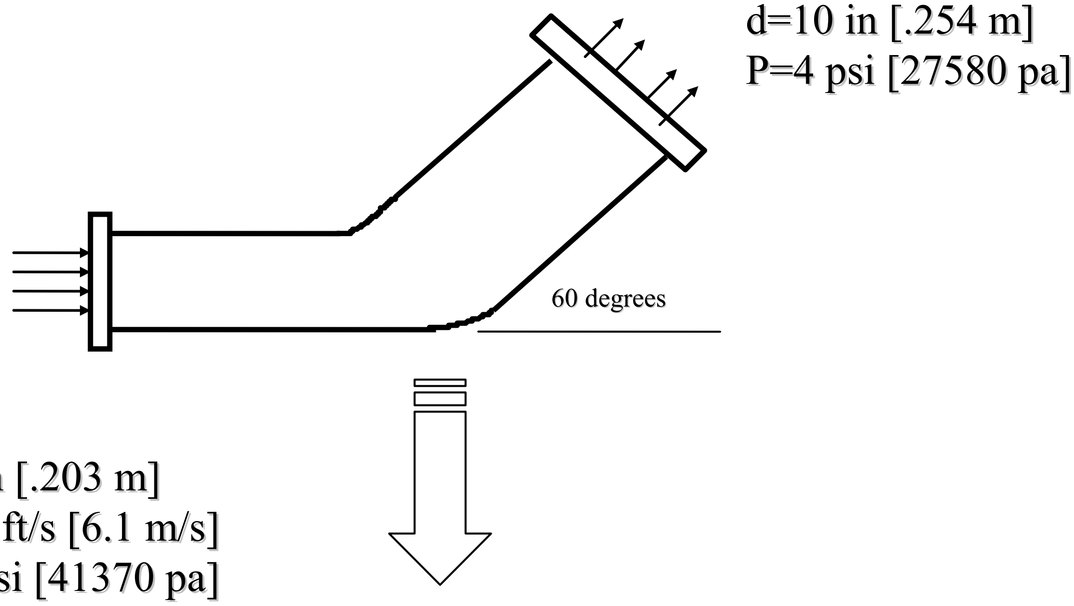
$$A_{in}V_{in} = A_{out}V_{out}$$

where A and V are the flow area and the flow velocity magnitudes.

Data:

Mass density of water is 1.94 slugs/ft^3
Mass density of water is 1000 kg/m^3

Example: Consider a horizontal pipe with the bend shown. Determine the vertical force exerted by the water on the bend support. The bend angle is 60 degrees.



Using conservation of mass

$$A_{in}V_{in} = A_{out}V_{out}$$

$$V_{out} = V_{in}\left(\frac{d_{in}}{d_{out}}\right)^2 = 20\left(\frac{8}{10}\right)^2 = 12.8 \quad ft/s$$

The mass density of water is about 1.94 slugs per cubic feet. The mass flow rate is:

$$\dot{m} = A\rho V = \frac{\pi d_{in}^2}{4}\rho V_{in}$$

$$\dot{m} = \frac{\pi}{4}(\frac{8}{12})^2(1.94)(20) = 13.5\, slugs/s$$

Momentum relationship (force needed to divert fluid)

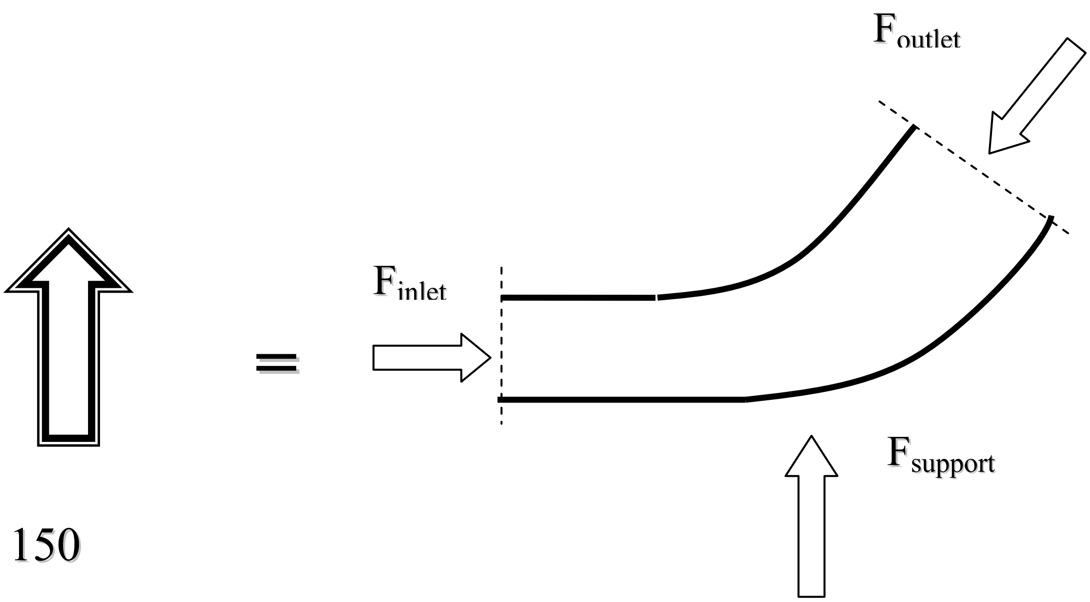
$$\vec{R} = \dot{m}(\vec{V}_{out} - \vec{V}_{in})$$

The vertical component is:

$$R_y = \dot{m}(V_{y,out} - V_{y,in})$$

$$R_y = 13.5(12.8 Sin 60 - 0) = 150\, lbs$$

This is the net force acting on water. Now we must consider the inlet and outlet pressures. The total resultant force is composed of three terms:



Equating the forces in the Y-direction:

$$R_y = F_{pipe} - F_{outlet} Sin(60)$$

Where

$$F_{out} = P_{outlet} A_{outlet} = 4(\frac{\pi(10)^2}{4}) = 314.16$$

Solving for the support force

$$150 = F_{\sup} - 314.16\sin(60)$$

$$F_{\sup} = 422 \quad lbs \quad \{1877\ N\}$$

The pressure difference increases the support force substantially.

In SI Units

$$V_{out} = V_{in}(\frac{d_{in}}{d_{out}})^2 = 6.1(\frac{.203}{.254})^2 = 3.9 \quad m/s$$

The mass density of water is about 1000 kg per cubic meters. The mass flow rate is:

$$\dot{m} = A\rho V = \frac{\pi d_{in}^2}{4}\rho V_{in}$$

$$\dot{m} = \frac{\pi}{4}(.203)^2(1000)(6.1) = 197.43\, kg/s$$

Momentum relationship (force needed to divert fluid)

$$R_y = \dot{m}(V_{y,out} - V_{y,in})$$

$$R_y = 197.43(3.9 Sin 60 - 0) = 666.8 \quad N$$

This is part (a) solution. Equating the forces in the Y-direction:

$$R_y = F_{pipe} - F_{outlet} Sin(60)$$

Where

$$F_{out} = P_{outlet} A_{outlet} = 27580(\frac{\pi(.254)^2}{4}) = 1397.5 \quad N$$

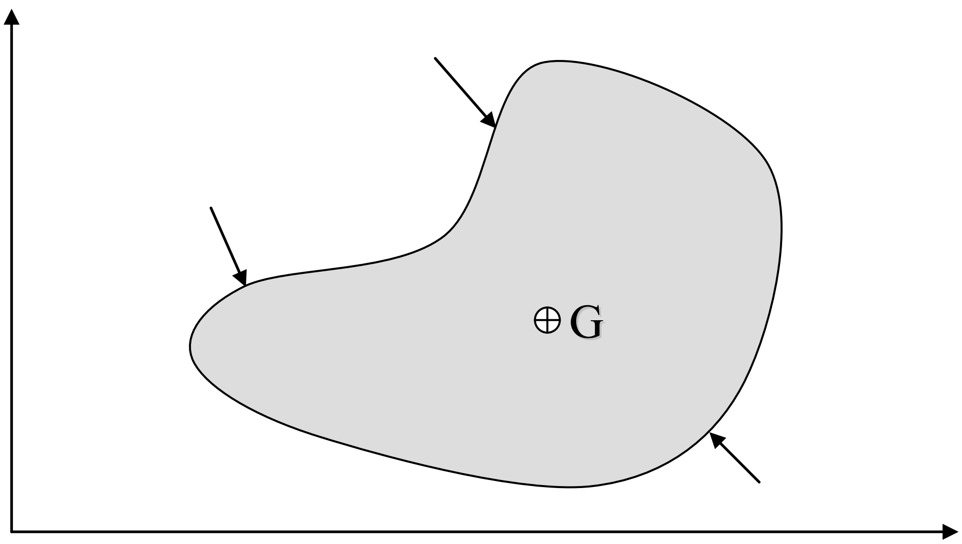
Solving for the support force

$$666.8 = F_{\sup} - 1397.5 \sin(60)$$

$$F_{\sup} = 1877 \quad N$$

Kinetics of Rigid Bodies

A rigid body is a special case of a system of particles in which the particles have a fixed rigid relationship with respect to each other. (no internal movements or internal work)



Force/Acceleration Formulation

$$\sum \vec{F}_{ext} = m_t \vec{a}_G$$

A rigid-body being one rigid object can also have an angular velocity ω and angular acceleration α. The motion of a rigid body is completely defined (at an instant) by its mass center acceleration as well as its angular acceleration.

Force Moment – Angular Acceleration Formulation

The angular impulse – angular momentum relationship for particle systems (which was not presented in its general form) takes a new and very useful form for rigid bodies:

$$\sum \vec{M}_{ext,O} = I_O \vec{\alpha}$$

Where point O is a fixed (non-accelerating) point. The constant I_O is called the *mass moment of inertia* of the rigid body about point O:

$$I_o = \int_M R_o^{\,2} dm$$

It is a measure of rotational inertia. The more mass or radius, the harder it is to spin the body. Alternatively, the formulation can also be applied with respect to the rigid body's center of mass:

$$\sum \vec{M}_{ext,G} = I_G \vec{\alpha}$$

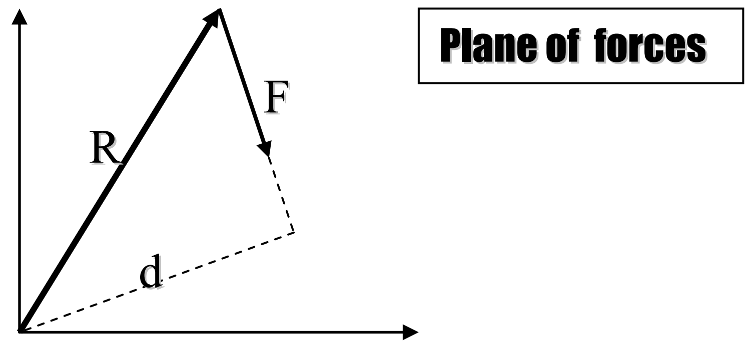
and

$$I_G = \int_M R_G^{\,2} dm$$

Relations for mass moments of inertias for common shapes can be found in dynamics book tables. This formula does not apply to any other points (other than O and G).

While the moment-angular acceleration formulation applies to 3D dynamic systems, most rigid body problems are planar allowing the moments of forces to be calculated using the familiar cross-product methods.

The magnitude of the moment vector is F.d where F is the



magnitude of the force vector and d is the normal distance. The direction of the moment vector is either out of the plane or into the plane. This direction is identified by a plus sign or a minus sign. The convention is to use the right-hand rule for postive or negative directions. For example the figure shows a moment going into the page according to the rule and therefore the moment value would be negative.

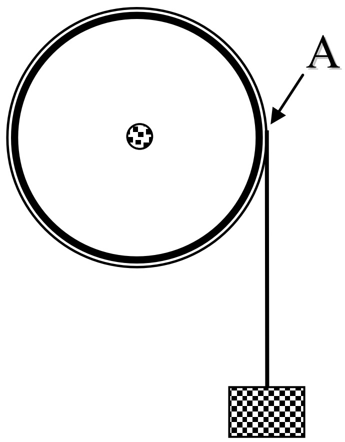
Parallel-Axis theorem

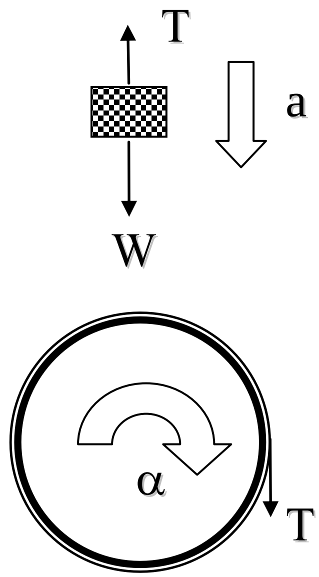
A mass moment of interias with respect to any axis can be easily calculated if the moment of inertia is known with respect to a parallel axis at CG. The relationship is:

$$I_O = I_G + md^2$$

where *d* is the distance between the parallel axes.

Example: A 120 lb (533.76 N) pulley (r =8" 0.2032 m) is released from rest lowering a box with a mass of 2 slugs (29.16 kg). During lowering the rope tension remains constant. What value is this rope tension?





Choosing downward to be positive and from the freebody diagram of the box:

$$\oplus\downarrow \quad \sum F = ma$$

$$W - T = ma$$

Where

W = 2(32.2) = 64.4 lbs and m = 2 slugs

Also from kinematics of motion:

$$a = r\alpha$$

Note that the direction of ***α*** must be consistent with **a**. The equation of motion for the mass becomes

$$64.4 - T = 2(\frac{8}{12})\alpha \qquad \text{(I)}$$

$$OR \quad T = 64.4 - 1.33\alpha$$

Now from the FBD of the pulley (ignoring pivot reaction forces as they are not needed for moment relationship)

$$\sum M_O = I_O\alpha \quad \Rightarrow T(\frac{8}{12}) = \frac{1}{2}(\frac{120}{32.2})(\frac{8}{12})^2\alpha$$

$$OR \quad T = 1.24\alpha \qquad \text{(II)}$$

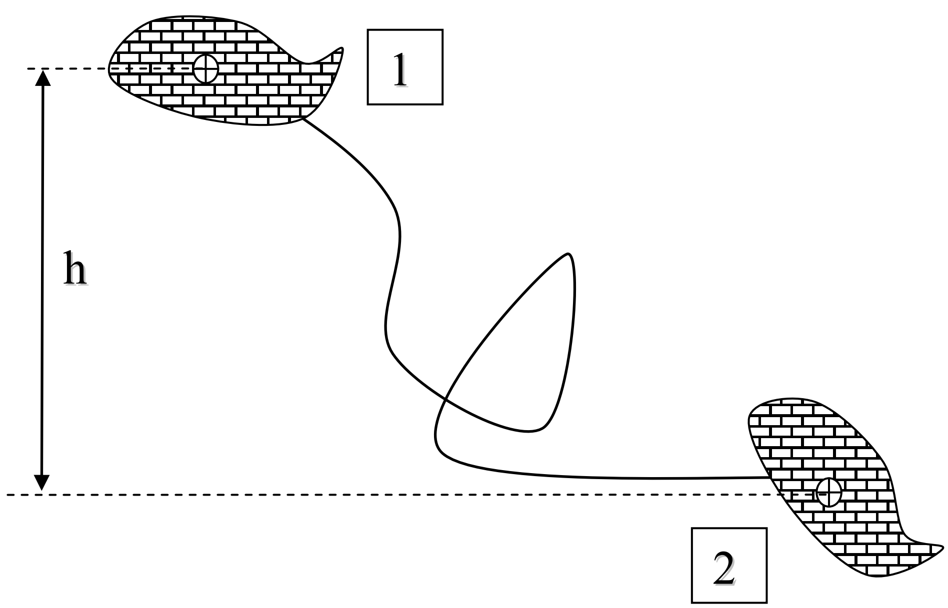
Note that the direction of T in the pulley FBD must be consistent with that in the box FBD. Setting (II) into (I)

T = 64.4 – 1.33*(T/1.24)

T=31.1 lbs

SI *T = 138.33 N*

Work and Kinetic Energy Formulation for Rigid Bodies



The energy relationship is the same as in particle systems:

$$\sum U_{i,1-2} = \Delta KE_{1-2} + \Delta PE_{1-2}$$

In this relation the work done on the system is due to external forces only (the work of all internal forces cancel out for single rigid bodies).

Kinetic Energy of a Rigid Body

$$T = \frac{1}{2} m V_G^2 + \frac{1}{2} I_G \omega^2$$

When the rigid body is pinned at a fixed point O and is only rotating, the kinetic energy is:

$$T = \frac{1}{2} I_O \omega^2$$

The change in potential energy due to gravitational force and spring forces remain as before:

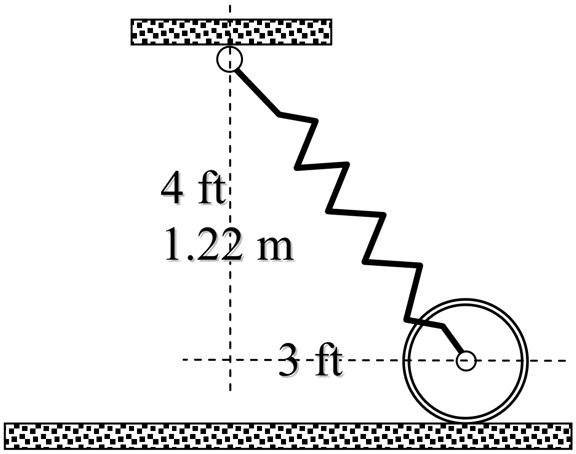
$$\Delta PE_{gravity} = mg(h_2 - h_1)$$

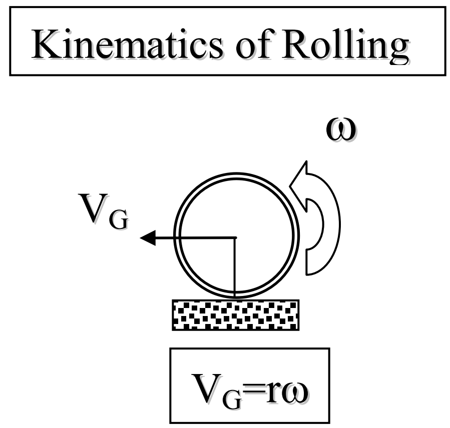
and

$$\Delta PE_{Elastic} = \frac{1}{2} k(\Delta S_2^2 - \Delta S_1^2)$$

Example

A uniform disk of raduis 0.75 ft (0.2286 m) and weight 30lbs (133.44 N) is pulled by a spring with K=2 lb/ft (29.16667 N/m). The free length of the spring is 1 ft (0.3048 m). The disk is released from rest at the position shown. The disk is going to roll without slipping. Find the angular velocity of the disk as it travels 3 ft(0.9144 m) toward left.





Change Kinetic Energy

$$T_1 = 0$$

$$T_2 = \frac{1}{2}mV_G^2 + \frac{1}{2}I_G\omega^2$$

$$T_2 = \frac{1}{2}(\frac{30}{32.2})(.75)^2(\omega)^2 + \frac{1}{2}\left[\frac{1}{2}(\frac{30}{32.2})(.75)^2\right]\omega^2 = 0.393\omega^2$$

In SI

$$T_2 = \frac{1}{2}(\frac{133.4}{9.8})(.23)^2(\omega)^2 + \frac{1}{2}\left[\frac{1}{2}(\frac{133.4}{9.8})(.23)^2\right]\omega^2 = 0.54\omega^2$$

Change in Potential Energy

$$U_1 = \frac{1}{2}k(5-1)^2 = \frac{1}{2}(2)(4)^2 = 16 \quad ft-lb$$

$$U_2 = \frac{1}{2}k(4-1)^2 = \frac{1}{2}(2)(3)^2 = 9 \quad ft-lb$$

In SI

$$U_1 = \frac{1}{2}(29.17)(1.52-.30)^2 = 21.7 \quad N-m$$
$$U_2 = \frac{1}{2}(29.17)(1.22-.30)^2 = 12.3 \quad N-m$$

Applying the energy formulation:

$$0.393\omega^2 = 7 \quad \Rightarrow \quad \omega = 4.22 \quad rad/\sec$$

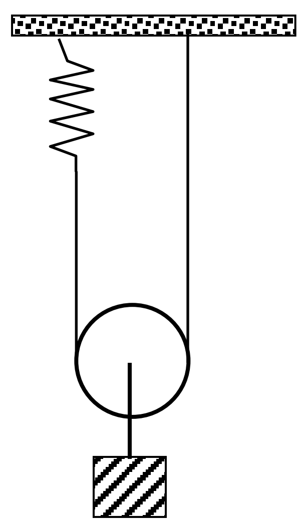
in SI

$$0.54\omega^2 = 9.4 \quad \Rightarrow \quad \omega = 4.17 \quad rad/\sec$$

The diference is due to truncation errors with SI units.

#D8 Solve the problem when a constant force of 1-lb [4.45 N] acts on the roller center tending to slow it down. Ans: 3.2 rad/s

#D9 The disk in the figure has a mass of 1 slugs [14.6 kg] and a raduis of gyration of 2.5 in. [0.063 m] and a raduis of 4 inches [0.1 m]. The block's mass is 2 slugs [29.2 kg]. The spring constant is 40 lb/ft [583 N/m] and the free length of the spring is 5 in. [0.127 m]. The system is released from rest when the spring is at free length. What is the velocity of the block after it falls 9 inches [.23 m].



Ans: V=4 ft/s [1.22 m/s] {in SI the ans. is slightly different}

Impulse-Momentum Formulation of Rigid Bodies

The linear Impulse of the resultant external force on a rigid body changes the linear momentum of the body as follows:

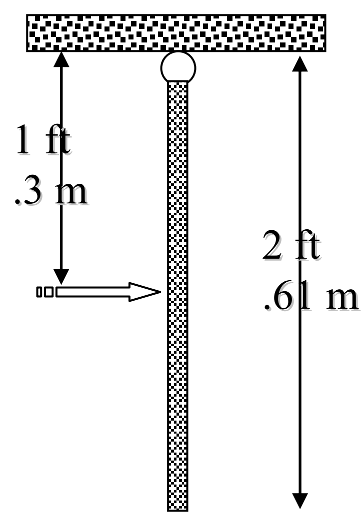
$$\int \vec{F}_{ext} dt = m(\vec{V}_{G,f} - \vec{V}_{G,i})$$

Also, in a planar motion, the change in the angular momentum of a rigid body with resprct to a fixed point O is equal to the angular impulse of the all external forces about O:

$$\int \sum M_{ext,O} dt = I_O(\omega_f - \omega_i)$$

It is easy to see that if we take derivatives of both sides with respect to time, we get $\Sigma M_{ext} = I\alpha$.

#D10 A 1-lb [4.45 N] uniform rod is struck by a bullet weighing 0.05 pounds(.2224 N). The bar's maximum angle of swing is measured to be 120 degrees. Estimate the velocity of the bullet in feet per second. Ignore friction. The bullet gets embedded in the rod.



Answer: 226 ft/sec [68.9 m/s]
(*44 ft/sec if you ignore the energy loss during impact!*)