Overview of Fourier Representation Properties

- Range of equations (finite versus infinite)
- Signal and transform classes (periodic versus nonperiodic)
- Linearity
- Time and frequency shifts
- Symmetry of transforms
- Transform relationship to signal symmetry
- Amplitude/phase representations
- Time and frequency scaling
- Convolution: nonperiodic versus periodic signals
- Multiplication
- Parseval’s theorem
- Duality
## Review of Signal Types

<table>
<thead>
<tr>
<th>Type</th>
<th>Series/Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>DT Periodic</td>
<td>DT Fourier Series</td>
</tr>
<tr>
<td>CT Periodic</td>
<td>CT Fourier Series</td>
</tr>
<tr>
<td>DT Nonperiodic</td>
<td>DT Fourier Transform</td>
</tr>
<tr>
<td>CT Nonperiodic</td>
<td>CT Fourier Transform</td>
</tr>
</tbody>
</table>

- We have discussed two categories of signals
  - Continuous-time & discrete-time
  - Periodic and nonperiodic

- Each combination of these properties has an *appropriate* transform

- Keep in mind
  - Periodic signals are power signals
  - The CTFT and DTFT usually only converge for energy signals
  - Can be applied to periodic and almost periodic signals if we allow the CTFT and DTFT to include impulses
Fourier Series & Transform Summary

**DTFS**

\[ x[n] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_o n} \]

\[ X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_o n} \]

**CTFS**

\[ x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t} \]

\[ X[k] = \frac{1}{T} \int_{-T}^{T} x(t) e^{-jk\omega_o t} \, dt \]

**DTFT**

\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\Omega n} \, d\Omega \]

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \]

**CTFT**

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} \, d\omega \]

\[ X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt \]

- Each transform has a **synthesis** and **analysis** equation
- Each transforms the signal (function of time) into a function of frequency
**Fourier Series & Transform Observations**

- *Periodic* signals can be represented as a *sum* of sinusoids
- *Nonperiodic* signals can be represented as an *integral* of sinusoids
- *DT signals*
  - Can be represented by a *finite frequency range*
  - Have transforms that are periodic
    \[
    X[k] = X[k + N] \quad X(e^{j\Omega}) = X(e^{j(\Omega + 2\pi)})
    \]
    - This is a consequence of \(e^{j\Omega n} = e^{j(\Omega \pm \ell 2\pi)n}\)
- *CT signals*
  - Must be represented by an *infinite frequency range*, in general
  - Have transforms that are nonperiodic, in general
    \[
    X[k] \neq X[k + N] \quad X(j\omega) \neq X(j(\omega + \omega_o))
    \]
    - This is because \(e^{j\omega_1 t} \neq e^{j\omega_2 t}\) unless \(\omega_1 = \omega_2\)
Notes on Terminology

• Many of the transforms share the same properties
• The following slides cover the primary properties of the transforms
• I will use the word *transform* generally to refer to both the Fourier series transforms and the Fourier transforms
• The book avoids this and uses the word *representation*
Property 1: Linearity

\[ a_1 x_1[n] + a_2 x_2[n] \begin{cases} \mathcal{FS} & \Rightarrow a_1 X_1[k] + a_2 X_2[k] \\ \mathcal{FT} & \Rightarrow a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega}) \end{cases} \]

\[ a_1 x_1(t) + a_2 x_2(t) \begin{cases} \mathcal{FS} & \Rightarrow a_1 X_1[k] + a_2 X_2[k] \\ \mathcal{FT} & \Rightarrow a_1 X_1(j\omega) + a_2 X_2(j\omega) \end{cases} \]

- All of the Fourier transforms are linear
- This follows directly from the linearity property of sums and integrals
- Note that for periodic signals, both components must have the same fundamental period
Example 1: Linearity

Derive one of the four linearity relationships given on the previous slide.
Application Example 1: Applying the Definition

Suppose two people sing into a microphone at the same time. Under what conditions is it possible to design a linear time-invariant filter that separates one of the voices? What characteristics would you want the filter to have?
Application Example 1: Workspace
Application Example 2: Applying the Definition

Suppose you obtain an old recording of a song that you like that is corrupted with a constant humming background noise. Is it possible to design an LTI filter that will eliminate the humm? What characteristics would you want the filter to have? How much would it distort the original signal?
Application Example 2: Workspace
Property 2: Time Shift

\[ x[n - n_o] \begin{cases} \mathcal{FS} & \rightarrow e^{-j\omega_o n_o} X[k] \\ \mathcal{FT} & \rightarrow e^{-j\omega n_o} X(e^{j\omega}) \end{cases} \]

\[ x(t - t_o) \begin{cases} \mathcal{FS} & \rightarrow e^{-j\omega_o t_o} X[k] \\ \mathcal{FT} & \rightarrow e^{-j\omega t_o} X(j\omega) \end{cases} \]

- What effect does a time shift have on even/odd symmetry in general?
- What effect does a time shift have on phase?
- Suppose the phase of the Fourier representation is zero at all frequencies. What is the effect of shifting the signal?
- What effect does a time shift have on amplitude?
Example 2: Time Shift

Derive one of the four time-shift relationships on the previous slide.
Example 2: Workspace
Application Example 3: Speaker Recognition

Suppose you are asked to design a security system that only allows a single authorized person to open a door. To confirm the person’s identity, they must press a button and speak a password.

- How would you design such a system?
- How would you account for the variable delay between the start of the signal (when they push the button) and when they speak the word?
- How would you prevent an intruder from entering even if they say the same password?
Application Example 3: Workspace
Application Example 3: Workspace
Property 3: Frequency Shift

\[
e^{jk_s\Omega_o n} x[n] \overset{FS}{\iff} X[k - k_s]
\]

\[
e^{j\Omega_s n} x[n] \overset{FT}{\iff} X(e^{j(\Omega - \Omega_s)})
\]

\[
e^{jk_s\omega_s t} x(t) \overset{FS}{\iff} X[k - k_s]
\]

\[
e^{j\omega_s t} x(t) \overset{FT}{\iff} X(j(\omega - \omega_s))
\]

- \(\Omega_s\) and \(\omega_s\) is the frequency of the modulating complex sinusoid
- For periodic signals the modulating complex sinusoid must be an integer multiple of the fundamental frequency, \(k_s\Omega_o\) (why?)
- What effect does multiplying (modulating) a signal with a complex sinusoid have?
Example 3: Frequency Shift

Derive one of the four frequency-shift relationships on the previous slide.
Example 3: Workspace
Application Example 4: Communications Channel

Suppose you wish to transmit a stereo signal over a single telephone line. You can model a telephone line as a lowpass filter with a 4 kHz passband.

- How can you combine the two signals into a single signal in such a way that you can extract the original signals?
- What must you compromise in combining the two signals into a single signal?
- What properties must the signals have in order to send them over the telephone line without loss?
Example 4: Workspace
Example 4: Workspace
Property 4: Transform Symmetry

\[ X[-k] = X^*[k] \quad X(e^{-j\Omega}) = X^*(e^{j\omega}) \quad X(-j\omega) = X^*(j\omega) \]

- Each transform can be used to synthesize the signal as a sum or integral of complex sinusoids
- The resulting signal is real
- Thus all the imaginary components must cancel (sum or integrate to zero)
- This results in the complex-conjugate symmetry of the coefficients
Example 4: Transform Symmetry

Derive one of the four complex-conjugate symmetry relationships listed on the previous slide given that the signal is real-valued.
Example 4: Workspace
Property 4: Transform Symmetry Continued

If the signal is real, then the transforms are complex-conjugate symmetric about the origin

\[ X[-k] = X^*[k] \quad X(e^{-j\Omega}) = X^*(e^{j\omega}) \quad X(-j\omega) = X^*(j\omega) \]

- This implies
  - The real part of the transforms are even
  - The imaginary part of the transforms are odd
  - The magnitude is even
  - The phase is odd

- When this symmetry is present, it is only necessary to consider and plot the transforms for positive frequencies
Property 5: Transform Relationship to Signal Symmetry

\[ x_e[n] = \frac{1}{2} (x[n] + x[-n]) \quad \quad x_e(t) = \frac{1}{2} [x(t) + x(-t)] \]
\[ x_o[n] = \frac{1}{2} (x[n] - x[-n]) \quad \quad x_o(t) = \frac{1}{2} [x(t) - x(-t)] \]
\[ x[n] = x_e[n] + x_o[n] \quad \quad x(t) = x_e(t) + x_o(t) \]

- Every signal can be expressed as a sum of even and odd signals
- *Even* signals can be represented as a sum or integral of *cosines*
- *Odd* signals can be represented as a sum or integral of *sines*
- The *real* part of the Fourier coefficients represent the coefficients of the *cosines*
- The *imaginary* part of the Fourier coefficients represent the coefficients of the *sines*
Property 5: Transform Relationship to Signal Symmetry

If the signal is real, then

\[ x_e[n] \begin{cases} \mathcal{FS} & \text{Re}\{X[k]\} \\ \mathcal{FT} & \text{Re}\{X(e^{j\Omega})\} \end{cases} \quad \begin{cases} \mathcal{FS} & \text{Im}\{X[k]\} \\ \mathcal{FT} & \text{Im}\{X(e^{j\Omega})\} \end{cases} \quad \begin{cases} x_o[n] \begin{cases} \mathcal{FS} & \text{Re}\{X[k]\} \\ \mathcal{FT} & \text{Re}\{X(j\omega)\} \end{cases} \\ x_o(t) \begin{cases} \mathcal{FS} & \text{Im}\{X[k]\} \\ \mathcal{FT} & \text{Im}\{X(j\omega)\} \end{cases} \end{cases} \]

- Even signals have Fourier coefficients that are real
- Odd signals have Fourier coefficients that are imaginary
- Signals without even or odd symmetry are complex
Example 5: Transform Relationship to Signal Symmetry

Derive one of the eight relationships listed on the previous slide given that the signal is real-valued. Hint: use the complex-conjugate symmetry of the transform.
Example 5: Workspace
Property 6: Amplitude-Phase Representation

If the signal is real, each of the synthesis equations can be written as

\[
x[n] = \sum_{k=0}^{[N/2]} A[k] \cos(k\Omega_0 n + \theta[k])
\]

\[
x[n] = \frac{1}{\pi} \int_{0}^{\pi} A(e^{j\Omega}) \cos (\Omega n + \theta(e^{j\Omega})) \, d\Omega
\]

\[
x(t) = \sum_{k=0}^{\infty} A[k] \cos(k\omega_0 t + \theta[k])
\]

\[
x(t) = \frac{1}{\pi} \int_{0}^{\infty} A(j\omega) \cos (\omega t + \theta(j\omega)) \, d\omega
\]

- The \textit{magnitude} of the Fourier coefficients represents the magnitude of a combination of cosine and sine at each frequency.
- \textit{Complex phase angle} of the coefficients represents the \textit{phase} of the combined cosine.
Example 6: Transform Relationship to Signal Symmetry

Derive one of the amplitude-phase forms listed on the previous slide given that the signal is real-valued. Hint: use the complex-conjugate symmetry of the transform.
Example 6: Workspace
Example 6: Workspace
Amplitude/Phase versus Real/Imaginary Plots

- Engineers often plot the Fourier transforms versus frequency as part of signal analysis.
- This can lead to insights that guide design.
- The transforms are complex-valued functions of frequency.
- Requires two plots to fully represent.
- Two obvious choices:
  - Rectangular coordinates: Real and imaginary components versus frequency.
  - Polar coordinates: Amplitude and phase versus frequency.
- Recall:
  - Real component corresponds to even component of signal.
  - Imaginary component corresponds to odd component of signal.
- But an even/odd signal decomposition, $x(t) = x_e(t) + x_o(t)$, is rarely helpful in practice.
**Application Example 5: Polar versus Rectangular**

Use the fast Fourier transform to estimate the CTFT over a range of frequencies from 0 to half the sampling rate, $f_s/2$, of a 60 ms segment of speech.

- What insights do you gain from studying the FT plots that you did not gain from studying a time-domain plot?
- What type of properties would a communications channel require to transmit this signal?
- What insights do you gain from studying the real, imaginary, amplitude, and phase plots.
Application Example 5: Polar versus Rectangular

Linus: Philosophy of Wet Suckers

Time (sec)
Application Example 5: Polar versus Rectangular

Real FT

Imaginary FT

Frequency (Hz)
Application Example 5: Polar versus Rectangular

- FT Magnitude
- FT Complex Phase (degrees)

Frequency (Hz):
0 500 1000 1500 2000 2500 3000 3500 4000 4500 5000 5500

FT Magnitude:
0 5 10 15 20

FT Complex Phase (degrees):
-200 -100 0 100 200
function [] = Speech();
close all;

[x,fs,nbits] = wavread('WetSuck.wav');

k = round(fs*1.6):round(fs*1.66); % Look at only 0.5 s
x = x(k);
x = x(k);
x = length(x);

figure;
FigureSet(1,'LTX');

nx = length(x);
t = (k-1)/fs;
h = plot(t,x,'b');
set(h,'LineWidth',0.6);
xrng = max(x)-min(x);
xlim([min(t) max(t)]);
ylim([min(x)-0.01*xrng max(x)+0.01*xrng]);
AxisLines;
xlabel('Time (sec)');
ylabel('');
title('Linus: Philosophy of Wet Suckers');
box off;
AxisSet(8);
print -depsc Speech;

X = fft(x,2^max([12,nextpow2(nx)]));
nX = length(X);
k = 1:floor((length(X)+1)/2);
f = (k-1)*(fs./(nX+1));

figure;
FigureSet(1,'LTX');

subplot(2,1,1);
h = plot(f,real(X(k)),'r');
```matlab
set(h,'LineWidth',0.6);
xlim([min(f) max(f)]);
ylim([1.05*min(real(X(k))) 1.05*max(real(X(k)))]);
set(gca,'XTick',[0:500:max(f)]);AxisLines;
box off;
ylabel('Real FT');

subplot(2,1,2);
  h = plot(f,imag(X(k)),'r');set(h,'LineWidth',0.6);
xlim([min(f) max(f)]);
ylim([1.05*min(imag(X(k))) 1.05*max(imag(X(k)))]);
set(gca,'XTick',[0:500:max(f)]);AxisLines;
box off;
ylabel('Imaginary FT');
xlabel('Frequency (Hz)');
AxisSet(6);
print -depsc SpeechFTRectangular;

figure;
FigureSet(1,'LTX');
subplot(2,1,1);
  h = plot(f,abs(X(k)),'r');set(h,'LineWidth',0.6);
xlim([min(f) max(f)]);
ylim([0 1.05*max(abs(X(k)))]);
set(gca,'XTick',[0:500:max(f)]);
box off;
ylabel('FT Magnitude');

subplot(2,1,2);
  h = plot(f,angle(X(k))*180/pi,'r');set(h,'LineWidth',0.6);
xlim([min(f) max(f)]);
ylim([-200 200]);
set(gca,'XTick',[0:500:max(f)]);AxisLines;
```
box off;
ylabel('FT Complex Phase (degrees)');
xlabel('Frequency (Hz)');
AxisSet(6);
print -depsc SpeechFTPolar;
Amplitude/Phase versus Real/Imaginary Plots Continued

- The most useful plot is amplitude (or squared amplitude) of the transform versus frequency
- Only need to plot transform for positive frequencies (why?)
- Examining the phase is rarely useful for signal analysis
  - Phase is sensitive to time-shift of signal
  - Phase is bounded: $-\pi \leq \theta \leq \pi$
- Phase is important for system analysis
  - Consider bode plots
  - Linearity of phase is an important consideration
  - More on this later
Property 7: Convolution of Nonperiodic Signals

\[ x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]

\[ x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau \]

- Convolution is defined differently for periodic and nonperiodic signals
- We will discuss convolution for nonperiodic signals first
- This is the convolution you are already familiar with
Example 7: Nonperiodic Continuous-Time

Derive either the DT or CT relationship between the convolution of two signals and their transforms

\[ x(t) * h(t) \overset{FT}{\leftrightarrow} X(j\omega)H(j\omega) \quad \text{or} \quad x[n] * h[n] \overset{FT}{\leftrightarrow} X(e^{j\omega})H(e^{j\omega}) \]
Example 7: Workspace
Property 7: Convolution Concepts

\[ x[n] * h[n] \xrightarrow{\mathcal{F}} X(e^{j\omega})H(e^{j\omega}) \quad x(t) * h(t) \xrightarrow{\mathcal{F}} X(j\omega)H(j\omega) \]

- This is very similar to what we used the Laplace transform for
- Key differences
  - Can be applied to two-sided signals
  - Cannot be applied to signals with infinite energy
  - Works in discrete-time as well as continuous-time
- Enables us to think of a system in terms of how it \textit{scales} and \textit{shifts the complex phase angle} of each complex sinusoid component of the signal
- This is very useful if we have a good notion of which components are present in a signal via the Fourier transform
Property 8: Convolution of Periodic Signals

\[ x(t) \ast z(t) = \int_{<T>} x(\tau)z(t - \tau) \, d\tau \quad x(t) \ast z(t) \xrightarrow{FS} T X[k]Z[k] \]

\[ x[n] \ast z[n] = \sum_{k=<N>} x[k]z[n - k] \quad x[n] \ast z[n] \xrightarrow{FS} N X[k]Z[k] \]

- Convolution of periodic signals occurs in signal analysis, but not system analysis.
- This is because stable systems do not have impulse responses that are periodic.
- Nonetheless, convolution is useful for signal analysis.
Example 8: Convolution of Periodic Signals

Derive either the DT or CT relationship between the convolution of two signals and their transforms

\[ x(t) \ast h(t) \xleftrightarrow{FS} TX[k]Z[k] \quad x[n] \ast h[n] \xleftrightarrow{FS} NX[k]Z[k] \]
Property 9: Differentiation and Integration

- These are listed in the book (Table 3.6)
- They are very similar to the corresponding Laplace transform properties
- In most cases, these are handled with other transforms
  - Differential equations are usually handled with the Laplace transform
  - Difference equations are handled with the $z$ transform
Property 10: Signal Multiplication

\[ x[n] \cdot z[n] \xrightarrow{FS} X[k] \ast Z[k] \]

\[ x[n] \cdot z[n] \xrightarrow{FT} \frac{1}{2\pi} X(e^{j\omega}) \ast Z(e^{j\omega}) \]

\[ x(t) \cdot z(t) \xrightarrow{FS} X[k] \ast Z[k] \]

\[ x(t) \cdot z(t) \xrightarrow{FT} \frac{1}{2\pi} X(j\omega) \ast Z(j\omega) \]

- Recall that convolution in the time domain is equivalent to multiplication in the frequency domain
- This shows that multiplication in the time domain is equivalent to convolution in the frequency domain
- Recall that the DT transforms are periodic and the CT transforms are non-periodic
- Naturally, it works out that the DT transforms have circular convolution and the CT transforms have ordinary convolution
Example 9: Signal Multiplication

Derive one of the four relationships on the previous slide.
Windowing

$$x_s(t) = w_T(t)x(t) = \begin{cases} w_T(t)x(t) & |t| \leq T \\ 0 & |t| > T \end{cases}$$

$$X_s(j\omega) = \int_{-T}^{T} w_T(t)x(t)e^{-j\omega t} \, dt$$

- To estimate the Fourier transform of a signal, modern equipment uses **digital signal processing** (DSP)
- This essentially means the signal is first sampled, and then processed
- Only a finite segment of the sample can be analyzed, $x_s(t)$
- Using this segment can be modelled as multiplying the original signal with a finite-duration window, $w(t)$
- This processing step is called **windowing**
- How does this affect the estimated spectrum of the signal?
Since only of a segment of the signal is available, the Fourier transform of the signal cannot be calculated exactly.

Windowing is necessary to estimate the spectrum from a segment of the signal with finite duration.

In the frequency domain, this filters or **blurs** the spectrum.
Example 5: Windowing

Your oscilloscopes in the lab must estimate the CT Fourier transform of a signal using a finite segment of the data. Calculate and plot the CT Fourier transform of a windowed sinusoid with a fundamental frequency of $f_0 = 10 \text{ Hz}$ for a range of window lengths. Select the sinusoidal phase and window alignment such that the signal has even symmetry (why?).

- How does the estimated spectrum compare to the true spectrum?
- What happens as the window length decreases towards zero?
- What happens as the window length increases towards infinity?
- What would the transform of an ideal window be?
- What are the disadvantages of using a rectangular window, $w(t) = p_T(t)$?
Example 10: Workspace

Hint: we found the Fourier transform of a pulse earlier:

\[ p(t) = u(t + T) - u(t - T) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases} \]

\[ P(j\omega) = 2T \frac{\sin(\omega T)}{\omega T} = 2T \text{sinc} \left( \frac{\omega T}{\pi} \right) \]
Example 10: Window and Windowed Segment

Window \( f_0 = 10.0 \text{ Hz} \) \( T=0.100 \text{ s} \)

Window \( w(t) \)

Windowed Signal \( y(t) = w(t) x_s(t) \)
Example 10: Window and Windowed Segment Transforms

Pulse Spectral Amplitude $f_o = 10.0$ Hz $T=0.100$ s

Frequency (rad/s)
Example 10: Window and Windowed Segment

Window $f_o = 10.0$ Hz $T=0.500$ s
Example 10: Window and Windowed Segment Transforms

Pulse Spectral Amplitude $f_0 = 10.0$ Hz $T = 0.500$ s

$W(j\omega)$

$Y(j\omega)$

Frequency (rad/s)
Example 10: Window and Windowed Segment

Window $f_o = 10.0 \text{ Hz}$ $T=1.000 \text{ s}$

Window $w(t)$

Windowed Signal $y(t) = w(t) x_s(t)$

Time (s)
Example 10: Window and Windowed Segment Transforms

Pulse Spectral Amplitude \( f_0 = 10.0 \text{ Hz} \) \( T=1.000 \text{ s} \)

- \( W(\omega) \)
- \( Y(\omega) \)

Frequency (rad/s)

\( Y(\omega) \) shows the frequency response of a windowed signal, while \( W(\omega) \) represents the window function itself.
Example 10: Window and Windowed Segment

Window \( f_0 = 10.0 \text{ Hz} \) \( T=2.000 \text{ s} \)

\[
\text{Windowed Signal } y(t) = w(t) x_s(t)
\]
Example 10: Window and Windowed Segment Transforms

Pulse Spectral Amplitude $f_o = 10.0 \text{ Hz} \ T=2.000 \text{ s}$

$W(j\omega)$

$Y(j\omega)$

Frequency (rad/s)
Example 10: Window and Windowed Segment

Window $f_0 = 10.0$ Hz $T=5.000$ s

Windowed Signal $y(t) = w(t) x_s(t)$

Time (s)
Example 10: Window and Windowed Segment Transforms

Pulse Spectral Amplitude  $f_o = 10.0$ Hz  $T=5.000$ s

$W(j\omega)$

$Y(j\omega)$

Frequency (rad/s)
Example 10: MATLAB Code
Example 10: MATLAB Code Continued
Property 11: Time & Frequency Scaling

\[ x[an] \xrightarrow{\mathcal{FT}} X(e^{j\Omega/a}) \]
\[ x(at) \xrightarrow{\mathcal{FS}} X[k] \]
\[ x(at) \xrightarrow{\mathcal{FT}} \frac{1}{|a|} X \left( \frac{j\omega}{a} \right) \]

- Scaling a signal in time inversely scales the spectrum by the same factor.
- Scaling a periodic signal shifts the spacing of the harmonic components, but the amplitudes are the same!
- In the DT case, \( a \) must be an integer (the signal is not defined at non-integer samples).
- This is equivalent to sampling only 1 out of every \( a \) values of \( x[n] \).
- DT periodic signals are tricky because the fundamental period of \( x[an] \) depends on how \( a \) and \( N \) are related.
Example 11: Time & Frequency Scaling

Derive one of the transform relationships on the previous slide.
Example 11: Workspace
Property 12: Parseval’s Theorem

\[
\begin{align*}
\text{DTFS} & \quad \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |X[k]|^2 \\
\text{DTFT} & \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi\rangle} |X(e^{j\omega})|^2 \, d\Omega \\
\text{CTFS} & \quad \frac{1}{T} \int_{\langle T\rangle} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X[k]|^2 \\
\text{CTFT} & \quad \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega
\end{align*}
\]

- The energy/power of the time domain signal is equal to the energy/power of the frequency domain transform!
Example 12: Parseval’s Theorem

Derive one of the four relationships on the previous slide.
Example 12: Workspace
Property 12: Parseval’s Theorem Comments

• Thus the magnitude of the spectrum squared is called
  – The **energy spectral density** for energy (nonperiodic) signals
  – The **power spectral density** for power (periodic) signals

• This is why the squared magnitude is often plotted versus frequency instead of just the magnitude

• This enables us to talk about what fraction of signal energy is contained within certain frequency ranges
Application Example 6: Parseval’s Theorem

Suppose you wish to transmit a speech signal through a channel with as little bandwidth as possible.

• What bandwidth is necessary if you wish to retain 95% of the signal energy?

• Which frequency range should you transmit?
Application Example 6: Parseval’s Theorem

Linus: Philosophy of Wet Suckers

Time (sec)

J. McNames  Portland State University  ECE 223  Fourier Properties  Ver. 1.11
Application Example 6: Parseval’s Theorem
Application Example 6: MATLAB Code
Application Example 6: MATLAB Code Continued
Property 13: Duality

\[ x(t) \overset{FT}{\leftrightarrow} X(j\omega) \quad X(jt) \overset{FT}{\leftrightarrow} 2\pi x(-t) \]
\[ x[n] \overset{FS}{\leftrightarrow} X[k] \quad X[n] \overset{FS}{\leftrightarrow} \frac{1}{N} x[-k] \]

• We have already seen many times there are equivalent relationships between the time and frequency domain

\[ x(t - t_o) \overset{FT}{\leftrightarrow} e^{-j\omega t_o} X(j\omega) \]
\[ \delta(t) \overset{FT}{\leftrightarrow} 1 \]
\[ p_T(t) \overset{FT}{\leftrightarrow} 2T \text{sinc} \left( \frac{\omega T}{\pi} \right) \]
\[ 2W \text{sinc} \left( \frac{tW}{\pi} \right) \overset{FT}{\leftrightarrow} 2\pi p_W(\omega) \]

• This occurs due to the similarity of the synthesis and analysis equations for the CTFT and DTFS
Example 13: Duality

Derive one of the four relationships on the previous slide.
Property 13: Duality Comments

\[ x(t) \overset{FT}{\longleftrightarrow} X(j\omega) \quad X(jt) \overset{FT}{\longleftrightarrow} 2\pi x(-t) \]

\[ x[n] \overset{FS}{\longleftrightarrow} X[k] \quad X[n] \overset{FS}{\longleftrightarrow} \frac{1}{N} x[-k] \]

- It occurs for only these two transforms because only these two have a signal and transform in the same signal class
  - CTFT: signal and transform are continuous and nonperiodic
  - DTFS: signal and transform or discrete and periodic
- There is also a duality relationship between the DTFT and the CTFS (see text for details), but this is less important
- Duality is useful primarily for calculating transforms
Fourier Properties Summary

• All four Fourier transforms share most properties

• These properties are useful for
  – Gaining insight that helps us understand how to interpret the transforms of signals (e.g. Parseval’s theorem)
  – Gives us tools and ideas for designing systems (e.g. modulation)