Overview of Discrete-Time Fourier Transform Topics

- Handy equations and limits
- Definition
- Low- and high- discrete-time frequencies
- Convergence issues
- DTFT of complex and real sinusoids
- Relationship to LTI systems
- DTFT of pulse signals
- DTFT of periodic signals
- Relationship to DT Fourier series
- Impulse trains in time and frequency
Handy Equations

\[ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1 \]

\[ \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \]

\[ \sum_{n=M}^{N-1} a^n = \frac{a^M - a^N}{1-a} \]

\[ \sum_{n=0}^{\infty} n a^n = \frac{a}{1-a}, \quad |a| < 1 \]

\[ \sum_{n=-N}^{N} a^n = \frac{a^{(N+0.5)} - a^{-(N+0.5)}}{a^{0.5} - a^{-0.5}} \]

You should be able to prove all of these.
Handy Limits

\[
\lim_{N \to \infty} \frac{\sin(\Omega(N \pm \frac{1}{2}))}{\sin(\Omega\frac{1}{2})} = +2\pi \sum_{\ell=-\infty}^{+\infty} \delta(\Omega - 2\pi\ell)
\]

\[
\lim_{N \to \infty} \frac{\cos(\Omega(N \pm \frac{1}{2}))}{\cos(\Omega\frac{1}{2})} = \pm2\pi \sum_{\ell=-\infty}^{+\infty} \delta(\Omega - \pi - 2\pi\ell)
\]

\[
\lim_{N \to \infty} \frac{\cos(\Omega(N \pm \frac{1}{2}))}{\sin(\Omega\frac{1}{2})} = 0
\]

\[
\lim_{N \to \infty} \frac{\sin(\Omega(N \pm \frac{1}{2}))}{\cos(\Omega\frac{1}{2})} = 0
\]

- First is roughly analogous to a sinc function
- All are periodic functions of frequency \( \Omega \) with fundamental period of \( 2\pi \)
Orthogonality Defined

Two non-periodic power signals $x_1[n]$ and $x_2[n]$ are orthogonal if and only if

$$\lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} x_1[n] x_2^*[n] = 0$$
Orthogonality of Complex Sinusoids

Consider two (possibly non-harmonic) complex sinusoids

\[ x_1[n] = e^{j\Omega_1 n} \quad x_2[n] = e^{j\Omega_2 n} \]

Are they orthogonal?

\[
\lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} x_1[n] x_2^*[n] = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} e^{j\Omega_1 n} e^{-j\Omega_2 n}
\]

\[
= \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} e^{j(\Omega_1 - \Omega_2) n}
\]

\[
= \begin{cases} 
1 & \Omega_1 - \Omega_2 = 2\pi \ell \\
0 & \text{Otherwise}
\end{cases}
\]
Importance of Orthogonality

Suppose that we know a signal is composed of a linear combination of non-harmonic complex sinusoids

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} \, d\Omega \]

How do we solve for the coefficients \( X(e^{j\Omega}) \)?

\[
\lim_{N \to \infty} \sum_{n=-N}^{N} x[n] e^{-j\Omega_o n} \\
= \lim_{N \to \infty} \sum_{n=-N}^{N} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} \, d\Omega \right] e^{-j\Omega_o n} \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \left[ \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} e^{j\Omega n} e^{-j\Omega_o n} \right] \, d\Omega
\]
\[
\begin{align*}
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \left[ \lim_{N \to \infty} \sum_{n=-N}^{N} e^{j(\Omega-\Omega_o)n} \right] d\Omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \left[ \lim_{N \to \infty} \frac{e^{j(\Omega-\Omega_o)(N+0.5)} - e^{-j(\Omega-\Omega_o)(N+0.5)}}{e^{j(\Omega-\Omega_o)0.5} - e^{-j(\Omega-\Omega_o)0.5}} \right] d\Omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) \left[ \lim_{N \to \infty} \frac{\sin[(\Omega - \Omega_o)(N + 0.5)]}{\sin[(\Omega - \Omega_o)0.5]} \right] d\Omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\Omega - \Omega_o \pm 2\pi\ell) d\Omega \\
&= \int_{-\pi}^{\pi} X(e^{j\Omega}) \delta(\Omega - \Omega_o) d\Omega \\
&= X(e^{j\Omega_o})
\end{align*}
\]
Definition

\[ F \{ x[n] \} = X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \]

\[ F^{-1} \{ X(e^{j\Omega}) \} = x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega \]

- Denote relationship as \( x[n] \overset{\mathcal{F}T}{\leftrightarrow} X(e^{j\Omega}) \)
- Why use this odd notation for the transform?
- Wouldn’t \( X(\Omega) \) be simpler than \( X(e^{j\Omega}) \)?
- Answer: this awkward notation is consistent with the \( z \)-transform

\[ X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \quad X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}} \]

- This also enables us to distinguish between the DT & CT Fourier transforms
Mean Squared Error

\[
\mathcal{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}
\]

\[
\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} \, d\Omega
\]

\[
\text{MSE} = \sum_{n=-\infty}^{+\infty} |x[n] - \hat{x}[n]|^2
\]

- Like the Fourier series, it can be shown that \(X(e^{j\Omega})\) minimizes the MSE over all possible functions of \(\Omega\)
- Like the DTFS, the error converges to zero
- Note: this isn’t in the text
Observations

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} \\
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega
\]

- Called the **analysis** and **synthesis** equations, respectively
- Recall that \(e^{j\Omega n} = e^{j(\Omega+\ell 2\pi)n}\), for any pair of integers \(\ell\) and \(n\)
- Thus, \(X(e^{j\Omega})\) is a periodic function of \(\Omega\) with a fundamental period of \(2\pi\)
- Unlike the DT Fourier series, the frequency \(\Omega\) is continuous
- Thus the DT **synthesis** integral can be taken over any continuous interval of length \(2\pi\)
\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} \]
\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega})e^{j\Omega n} \, d\Omega \]

- \( X(e^{j\Omega}) \) describes the frequency content of the signal \( x[n] \)
- \( x[n] \) can be thought of as being composed of a continuum of frequencies
- \( X(e^{j\Omega}) \) represents the density of the component at frequency \( \Omega \)
Equivalence of Discrete–Time Harmonics
function [] = Harmonics();
close all;

n = -10:10;
t = -10:0.01:10;

w = [0:0.2:1]*pi;
 nw = length(w);

FigureSet(1,'LTX');
for cnt = 1:length(w),
 subplot(nw,1,cnt);
   h = plot([min(t) max(t)],[0 0],'k:',t,cos(t*w(cnt)),'b',t,cos(t*(w(cnt)+2*pi)),'r');
   hold on;
   h = stem(n,cos(n*w(cnt)));set(h(1),'Marker','.');set(h(1),'MarkerSize',5);
   set(h,'Color','k');
   hold off;
   ylabel(sprintf('%3.1f \pi',w(cnt)/pi));ylim([-1.05 1.05]);box off;
   if cnt==1,
     title('Equivalence of Discrete-Time Harmonics');
     end;
   if cnt~=nw,
     set(gca,'XTickLabel','');
     end;
 AxisSet(8);print -depsc Harmonics;

Discrete-Time Frequency Concepts

- Recall that $e^{j(\Omega + \ell 2\pi)n} = e^{j\Omega n}$
- If seemingly very high-frequency discrete-time signals, $\cos((\Omega + \ell 2\pi)n)$, are equal to low-frequency discrete-time signals, $\cos(\Omega n)$, what does low- and high-frequency mean in discrete-time?
- Note that the units of $\Omega$ are radians per sample
- A sinusoid with a frequency of 0.1 radians per sample is the same as one with a frequency of $(0.1 + 2\pi)$ radians per sample
- Recall that $\cos(\pi n) = (-1)^n$
- No DT signal can oscillate “faster” between two samples
- No DT signal can oscillate “slower” than 0 radians per sample
- Thus
  - $\Omega = \pi = \ell(\pi + 2\pi)$ is the highest perceivable DT frequency
  - $\Omega = 0 = \ell(2\pi)$ is the lowest perceivable frequency
• Low frequencies are those that are near 0
• High frequencies are those near $\pm \pi$
• Intermediate frequencies are those in between
• Note that the highest frequency, $\pi$ radians per sample is equal to 0.5 cycles per sample
• We will encounter this concept again when we discuss sampling
Example 4: Unit Impulse

Find the Fourier transform of $x[n] = \delta[n]$. 
Convergence

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}
\]

\[
x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega})e^{j\Omega n} \, d\Omega
\]

- Sufficient conditions for the convergence of the discrete-time Fourier transform of a bounded discrete-time signal: (any one of the following are sufficient)
  - Finite duration: There exists an \( N \) such that \( x[n] = 0 \) for \( |n| > N \)
  - Absolutely summable: \( \sum_{n=-\infty}^{\infty} |x[n]| < \infty \)
  - Finite energy: \( \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \)

- The synthesis equation always converges
- There is no Gibb’s phenomenon in the time domain
Example 5: Inverse of Impulse Train

Sketch the following impulse train and find the inverse Fourier transform.

\[ X(e^{j\Omega}) = 2\pi \sum_{\ell=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi\ell) \]
Example 5: Workspace
Example 6: Constant

Find the Fourier transform of $x[n] = 1$. 
Example 6: Workspace
Convergence Revisited

\[ X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} \quad x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega})e^{j\Omega n} \, d\Omega \]

- The DTFT is said to converge if \( X(e^{j\Omega}) \) is finite for all \( \Omega \)
- We’ve now seen two examples where the DTFT was infinite at specific frequencies
- In these cases the DTFT didn’t converge
- But we were able to represent the transform with impulses \( \delta(\Omega - \Omega_0) \) anyway
- The use of impulses enables us to represent signals that contain periodic elements including constants and sinusoids
Fourier Transform & Transfer Functions

\[ Y(e^{j\Omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\Omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\Omega k}} X(e^{j\Omega}) = H(e^{j\Omega}) X(e^{j\Omega}) \]

- The time-domain relationship of \( y[n] \) and \( x[n] \) can be complicated.
- In the frequency domain, the relationship of \( Y(e^{j\Omega}) \) to \( X(e^{j\Omega}) \) of LTI systems described by difference equations simplifies to a rational function of \( e^{j\Omega} \).
- The numerator/denominator sums are not Fourier series.
- For real systems, \( H(e^{j\Omega}) \) is usually a rational ratio of two polynomials.
- \( H(e^{j\Omega}) \) is the discrete-time transfer function.
- Specifically, the transfer function of an LTI system can be defined as the ratio of \( Y(e^{j\Omega}) \) to \( X(e^{j\Omega}) \).
- Same story as the continuous-time case.
Example 7: Relationship to DT LTI Systems

Suppose the impulse response $h[n]$ is known for an LTI DT system. Derive the relationship between a sinusoidal input signal and the output of the system.
Example 7: Workspace
Example 8: Decaying Exponential

Find the Fourier transform of \( h[n] = a^n u[n] \) where \(|a| < 1\). Sketch the transform over a range of \(-3\pi\) to \(3\pi\) for \(a = 0.5\) and \(a = -0.5\).
Example 8: Workspace
Example 9: First-Order Filter $a = 0.5$

Fourier Transform of $(0.5)^n u[n]$
Example 10: First-Order Filter \( a = -0.5 \)

The Fourier Transform of \((-0.5)^n u[n]\)

- Magnitude: \( |X(e^{j\omega})| \)
- Phase: \( \angle X(e^{j\omega}) \)

Frequency (rad/sample)
%function [] = FirstOrder();
close all;

w = -3*pi:6*pi/1000:3*pi;
n = -8:8;

figure;
FigureSet(1,'LTX');
a = 0.5;
x = (a.^n).*(n>=0);
X = 1./((1-a*exp(-j*w));
Y = 1./(exp(-j*w/2).*2.*sin(w/2));
subplot(3,1,1);
    h = stem(n,x,'b');
    set(h(1),'MarkerFaceColor','b');
    set(h(1),'LineWidth',0.001);
    set(h(1),'MarkerSize',2);
    set(h(2),'LineWidth',0.6);
    set(h(3),'Visible','Off');
ylabel('x[n]');
title(sprintf('Fourier Transform of (%3.1f)^n u[n]',a));
xlim([min(n) max(n)]);
ylim([-0.55 1.06]);
box off;
AxisLines;
subplot(3,1,2);
    h = plot(w,abs(X),'r');
    set(h,'LineWidth',0.6);
    ylabel('|X(e^{j\omega})|');
xlim([-3*pi 3*pi]);
ylim([0 2.2]);
set(gca,'XTick',[-3:3]*pi);
box off;
grid on;
AxisLines;
subplot(3,1,3);
    h = plot(w,angle(X)*180/pi,'r');
    set(h,'LineWidth',0.6);
    ylabel('\angle X(e^{-j\omega}) (\degree)');
    xlabel('Frequency (rad/sample)');
    xlim([-3*pi 3*pi]);
    set(gca,'XTick',[-3:3]*pi);
box off;
grid on;
AxisLines;
AxisSet(8);
print -depsc FirstOrderLowpass;

figure;
FigureSet(1,'LTX');
a = -0.5;
x = (a.^n).*(n>=0);
X = 1./(1-a*exp(-j*w));
subplot(3,1,1);
    h = stem(n,x,'b');
    set(h(1),'MarkerFaceColor','b');
    set(h(1),'LineWidth',0.001);
    set(h(2),'LineWidth',0.6);
    set(h(3),'Visible','Off');
    ylabel('x[n]');
title(sprintf('Fourier Transform of (%3.1f)^n u[n]',a));
    xlim([min(n)-0.5 max(n)+0.5]);
    ylim([-0.55 1.06]);
    box off;
    AxisLines;
subplot(3,1,2);
    h = plot(w,abs(X),'r');
    set(h,'LineWidth',0.6);
    ylabel('|X(e^{-j\omega})|');
    xlim([-3*pi 3*pi]);
ylim([0 2.2]);
set(gca,'XTick',[-3:3]*pi);
box off;
grid on;
AxisLines;
subplot(3,1,3);
h = plot(w,angle(X)*180/pi,'r');
set(h,'LineWidth',0.6);
ylabel('$\angle X(e^{j\omega})$ (°)');
xlabel('Frequency (rad/sample)');
xlim([-3*pi 3*pi]);
set(gca,'XTick',[-3:3]*pi);
box off;
grid on;
AxisLines;
AxisSet(8);
print -depsc FirstOrderHighpass;
Example 11: Pulse

Find the Fourier transform of the following pulse signal. Sketch the transform over a range of $-3\pi$ to $3\pi$ for $N = 5, 10, \& 100$ and $N = 8$.

\[ p_N[n] = \begin{cases} 
1 & |n| \leq N \\
0 & |n| > N 
\end{cases} \]
Example 11: Workspace
Example 12: Pulse Transform for $N = 5$

Fourier Transform of $p_5[n]$
Example 13: Pulse Transform for $N = 10$

Fourier Transform of $p_{10}[n]$
Example 14: Pulse Transform for $N = 100$

**Fourier Transform of $p_{100}[n]$**

- $P(e^{j\omega})$
- $\omega$ values: $-9.4248, -6.2832, -3.1416, 0, 3.1416, 6.2832, 9.4248$
Example 14: MATLAB Code

```
function [] = Pulse();
close all;

s = 1e-4;
w = s:s:3*pi;
w = [-w(length(w):-1:1) w];

N = [5 10 100];
for c1=1:length(N),
    figure;
    FigureSet(1,'LTX');
    X = sin(w*(N(c1)+0.5))./sin(w/2);
    h = plot(w,X,'LineWidth',0.4);
    ylabel('P(e^{j\omega})');
    title(sprintf('Fourier Transform of p_{%d}[n]',N(c1)));
    xlim([-3*pi 3*pi]);
    ylim(1.05*[min(X) max(X)]);
    set(gca,'XTick',[-3:3]*pi);
    box off;
    grid on;
    AxisSet(8);
    print('-depsc',sprintf('Pulse%03d',N(c1)));
end;
```
Example 15: Periodic Discrete Time Signals

Find the Fourier transform of the following periodic signal.

\[ x[n] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_0 n} \]

where \( \Omega = \frac{2\pi}{N} \) for some integer \( N \).
Example 15: Workspace
Example 16: Relationship to Fourier Series

Suppose that we have a periodic signal $\tilde{x}[n]$ with fundamental period $N$. Define the truncated signal $x[n]$ as follows.

$$x[n] = \begin{cases} 
\tilde{x}[n] & n_0 + 1 \leq n \leq n_0 + N \\
0 & \text{otherwise}
\end{cases}$$

Determine how the Fourier transform of $x[n]$ is related to the discrete-time Fourier series coefficients of $\tilde{x}[n]$. Recall that

$$\tilde{X}[k] = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$
Example 16: Workspace
Summary of Key Concepts

- \( X(e^{j\Omega}) \) is a periodic function of \( \Omega \) with a fundamental period of \( \frac{2\pi}{2\pi} \)
- Two discrete-time complex exponentials with frequencies that differ by a multiple of \( 2\pi \) are equal: \( e^{j\Omega n} = e^{j(\Omega + \ell 2\pi)n} \)
- The highest perceivable discrete-time frequency is \( \pi \) radians per sample
- The lowest perceivable discrete-time frequency is \( 0 \) radians per sample
- The analysis equation may or may not converge, the synthesis equation always converges
- Quasi-periodic signals have a DTFT that consists of impulses at multiples of the fundamental frequency
- The DTFT enables us to analyze and understand systems described by difference-equations more thoroughly