Overview of Complex Sinusoids Topics

- Eigenfunction & eigenvalues of LTI systems
- Understanding complex sinusoids
- Four classes of signals
- Periodic signals
- CT & DT Exponential harmonics

Complex Sinusoids

- A **complex sinusoid** is defined as

$$x(t) = Ae^{j\omega t} = A\left[\cos(\omega t) + j\sin(\omega t)\right]$$

- These are a special case of **complex exponentials**

$$x(t) = Ae^{st} = Ae^{\alpha t}\left[\cos(\omega t) + j\sin(\omega t)\right]$$

when $\alpha = 0$

Review: Signals as Impulses

- Fourier series is used for signal decomposition
- In ECE 222 we decomposed signals into sums and of impulses

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,\mathrm{d}\tau \qquad x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

- We then used the LTI properties (linearity and time invariance) to solve for the output as a sum of the impulse responses

Review of Energy and Power Signals

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \qquad P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- A signal is an **energy signal** if $E_\infty < \infty$
- A signal is a **power signal** if $0 < P_\infty < \infty$
- Most signals are either energy signals or power signals
- A signal cannot be both

Orthogonalality

A pair of real-valued energy signals are orthogonal if

$$0 = \int_{-\infty}^{\infty} x_1(t) x_2^*(t) \, \mathrm{d}t$$

$$0 = \sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n]$$

A pair of real-valued power signals are orthogonal if

$$0 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) x_2^*(t) \, \mathrm{d}t$$

$$0 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_1[n] x_2^*[n]$$

Examples of Orthogonal Signals

Like impulses, complex sinusoids are special

- Impulses are **orthogonal** to one another

$$\int_{-\infty}^{\infty} \delta(t - t_1)\delta(t - t_2)\,\mathrm{d}t = 0 \text{ for } t_1 \neq t_2$$

$$\sum_{n=-\infty}^{\infty} \delta[n - n_1]\delta[n - n_2] = 0 \text{ for } n_1 \neq n_2$$

- Complex sinusoids are also **orthogonal** to one another

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{e}^{j\omega_1 t}\,\mathrm{e}^{j\omega_2 t}\,\mathrm{d}t = 0 \text{ for } \omega_1 \neq \omega_2$$

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \mathrm{e}^{j\Omega_1 n}\,\mathrm{e}^{j\Omega_2 n} = 0 \text{ for } \Omega_1 \neq \Omega_2$$

Importance of Orthogonality

- Orthogonal basis functions (e.g., complex sinusoids, shifted impulses) enable us to decompose signals into distinct (orthogonal) components
- More later

Eigenfunctions & Eigenvalues

- There are other basic signals that are also orthogonal
- But exponentials have another special property:
- You may be familiar with eigenvectors & eigenvalues for matrices
- There is a related concept for LTI systems
- Any signal $x(t)$ or $x[n]$ that is only scaled when passed through a system is called an **eigenfunction** of the system
  - $y(t) = x(t) * h(t) = c\,x(t)$
  - $y[n] = x[n] * h[n] = c\,x[n]$
- The scaling constant c is called the system's **eigenvalue**
- Complex exponentials are eigenfunctions of LTI systems
- *Complex sinusoids are the only eigenfunctions of LTI systems that have finite power!*

CT LTI System Response to Complex Exponentials

Let $x(t) = \mathrm{e}^{j\omega t}$. Then

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, \mathrm{e}^{j\omega(t-\tau)}\, \mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, \mathrm{e}^{j\omega t}\mathrm{e}^{-j\omega\tau}\, \mathrm{d}\tau \\
&= \mathrm{e}^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-j\omega\tau}\, \mathrm{d}\tau \\
&= \mathrm{e}^{j\omega t} H(j\omega)
\end{aligned}$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\mathrm{e}^{-j\omega\tau}\, \mathrm{d}\tau$$

DT LTI System Response to Complex Exponentials

Let $x[n] = \mathrm{e}^{j\Omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]\, \mathrm{e}^{j\Omega(n-k)} \\ &= \sum_{k=-\infty}^{\infty} h[k]\, \mathrm{e}^{j\Omega n} \mathrm{e}^{-j\Omega k} \\ &= \mathrm{e}^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]\, \mathrm{e}^{-j\Omega k} \\ &= \mathrm{e}^{j\Omega n} H(\mathrm{e}^{j\Omega}) \end{aligned}$$

where

$$H(\mathrm{e}^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]\, \mathrm{e}^{-j\Omega k}$$

Eigenfunctions & Eigenvalues

- Thus, complex sinusoids are eigenfunctions of LTI systems

$$e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}$$
$$e^{j\Omega n} \rightarrow H(e^{j\Omega n})e^{j\Omega n}$$

- The Fourier transform of the impulse response are the eigenvalues

$$H(j\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$
$$H(e^{j\Omega}) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]\,e^{-j\Omega n}$$

- $H(j\omega)$ and $H(e^{j\Omega})$ are functions of frequency
- Called the **frequency response** of the system

Example 1: Real and Complex Sinusoids

Are real sinusoids eigenfunctions of LTI systems?

Magnitude and Phase Response

$$H(j\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$$

$$H(e^{j\Omega}) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] \, e^{-j\Omega n}$$

- Both the eigenfunctions and eigenvalues of LTI systems are complex-valued
- $|H(j\omega)|$ and $|H(e^{j\Omega})|$ are called the **magnitude response**
- The complex phase angle of $H(j\omega)$ and $H(e^{j\Omega})$
  - Called the **phase response**
  - Denoted as $\arg H(j\omega)$ in the text
  - This is not the same as $\arctan\left\{\frac{\mathrm{Im}\{H(j\omega)\}}{\mathrm{Re}\{H(j\omega)\}}\right\}$, in general

Magnitude and Phase Response Continued

$$e^{j\omega t} \rightarrow |H(j\omega)|e^{j \arg H(j\omega)} e^{j\omega t} = |H(j\omega)|e^{j(\omega t + \arg H(j\omega))}$$

$$e^{j\Omega n} \rightarrow |H(e^{j\Omega})|e^{j \arg H(e^{j\Omega})} e^{j\Omega n} = |H(e^{j\Omega})|e^{j(\omega t + \arg H(e^{j\Omega}))}$$

Thus, if a complex sinusoid is applied at the input to an LTI system

- The system scales the amplitude by $|H(\cdot)|$
- The system changes the phase by $\arg H(\cdot)$

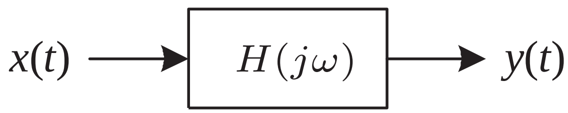
Sums of Complex Exponentials

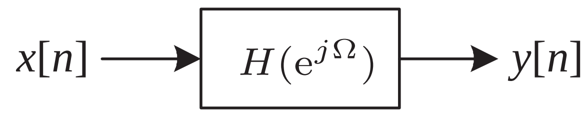
- Fourier series represent signals as sums (or integrals) of complex sinusoids

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{jk\omega_0 t} \qquad x[n] = \sum_{k=0}^{N_0-1} a_k \mathrm{e}^{jk\Omega_0 n}$$

- Since the signals are real, the imaginary portions of the complex exponentials must cancel out to zero
- This decomposition makes it particularly easy to solve for and analyze the system output
- Sums of complex sinusoids can only represent periodic and nearly periodic signals
- Integrals of complex sinusoids are more general

Complex Exponential Sums





By linearity and time-invariance (LTI),

$$x(t) = \sum_k a_k \mathrm{e}^{j\omega_k t} \qquad \rightarrow \qquad y(t) = \sum_k a_k H(j\omega_k) \mathrm{e}^{j\omega_k t}$$

$$x[n] = \sum_k a_k \mathrm{e}^{j\Omega_k n} \qquad \rightarrow \qquad y[n] = \sum_k a_k H(\mathrm{e}^{j\Omega_k}) \mathrm{e}^{j\Omega_k n}$$

- Thus if the input signal can be expressed as a sum of complex sinusoids, so can the output of the LTI system
- But what types of signals can be represented in this form?
- Virtually all of the (periodic) signals that we are interested in!
- Important and interesting idea

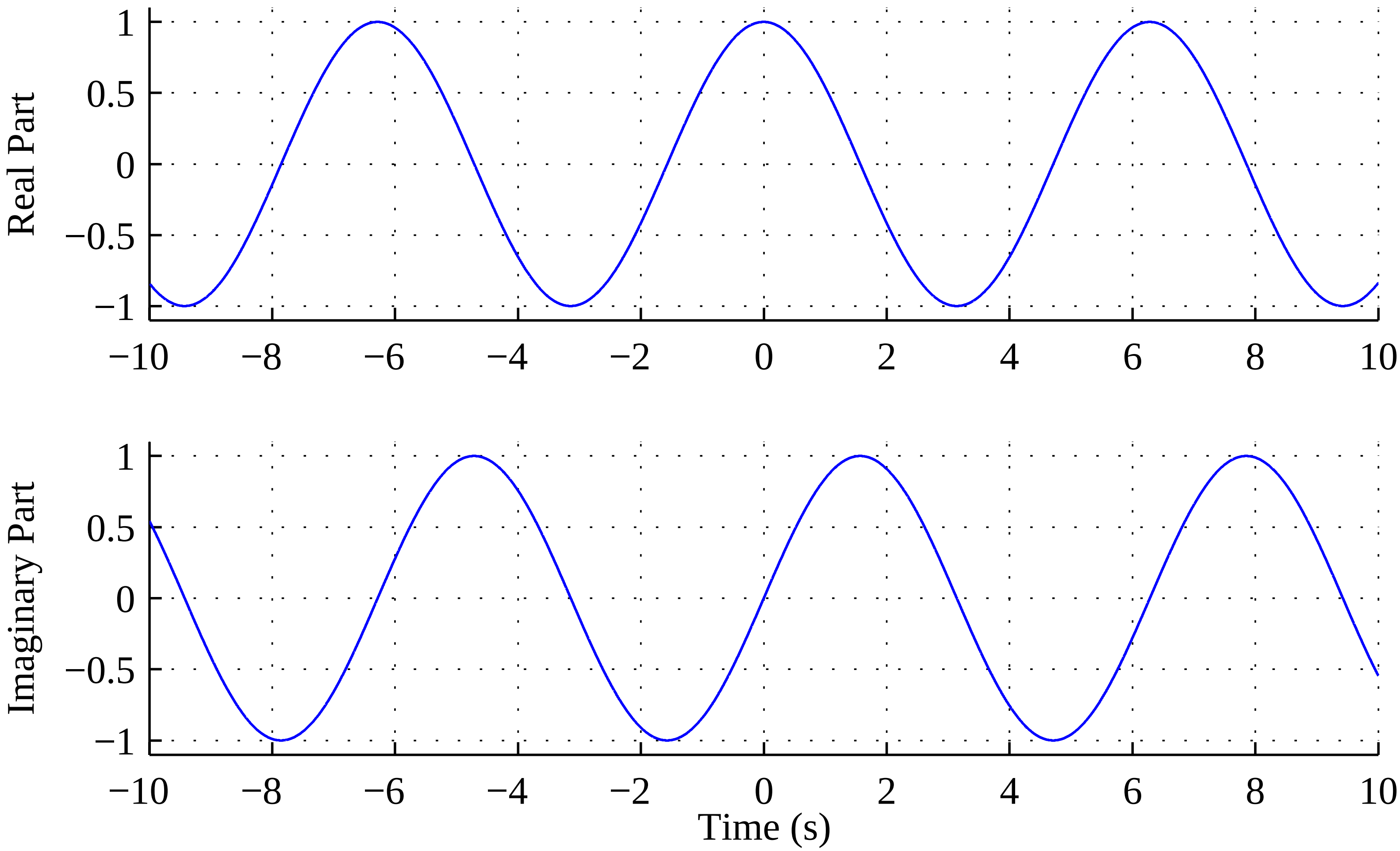
Continuous-Time Signal Intuition

$$x(t) = \sum_k a_k \mathrm{e}^{j\omega_k t} \qquad \rightarrow \qquad y(t) = \sum_k a_k H(j\omega_k) \mathrm{e}^{j\omega_k t}$$

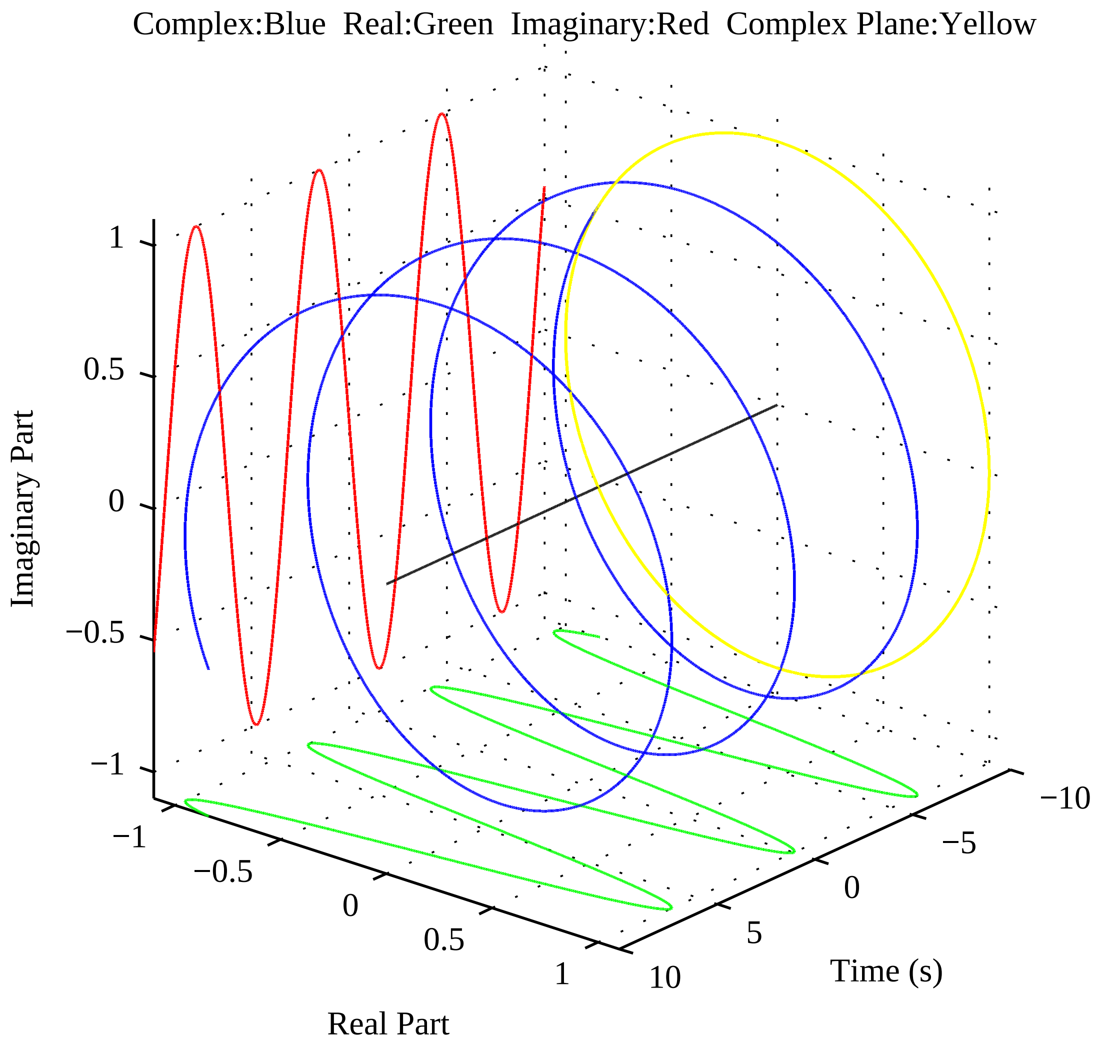
- Fourier transforms represent signals as sums (or integrals) of complex sinusoids
- It is therefore worthwhile to understand complex sinusoids as thoroughly as possible

$$x(t) = \mathrm{e}^{st}\Big|_{s=j\omega} = \mathrm{e}^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Example 2: $x(t) = \mathrm{e}^{jt}$



Example 2: $x(t) = \mathrm{e}^{jt}$



Example 2: MATLAB Code

```
w  = j*1;
fs = 500;                 % Sample rate (Hz)
t  = -10:1/fs:10;         % Time index (s)
y  = exp(w*t);
N  = length(t);
subplot(2,1,1);
   h = plot(t,real(y));
   box off;
   grid on;
   ylim([-1.1 1.1]);
   ylabel('Real Part');
subplot(2,1,2);
   h = plot(t,imag(y));
   box off;
   grid on;
   ylim([-1.1 1.1]);
   xlabel('Time (s)');
   ylabel('Imaginary Part');
```

Example 2: MATLAB Code Continued

```
figure;
h = plot3(t,zeros(1,N),zeros(1,N),'k');
hold on;
   h  = plot3(t,imag(y),real(y),'b');
   h = plot3(t,1.1*ones(size(t)),real(y),'r');
   h = plot3(t,imag(y),-1.1*ones(size(t)),'g');
   hold off;
grid on;
ylabel('Imaginary Part');
zlabel('Real Part');
title('Complex:Blue  Real:Red  Imaginary:Green');
axis([min(t) max(t) -1.1 1.1 -1.1 1.1]);
view(27.5,22);
```

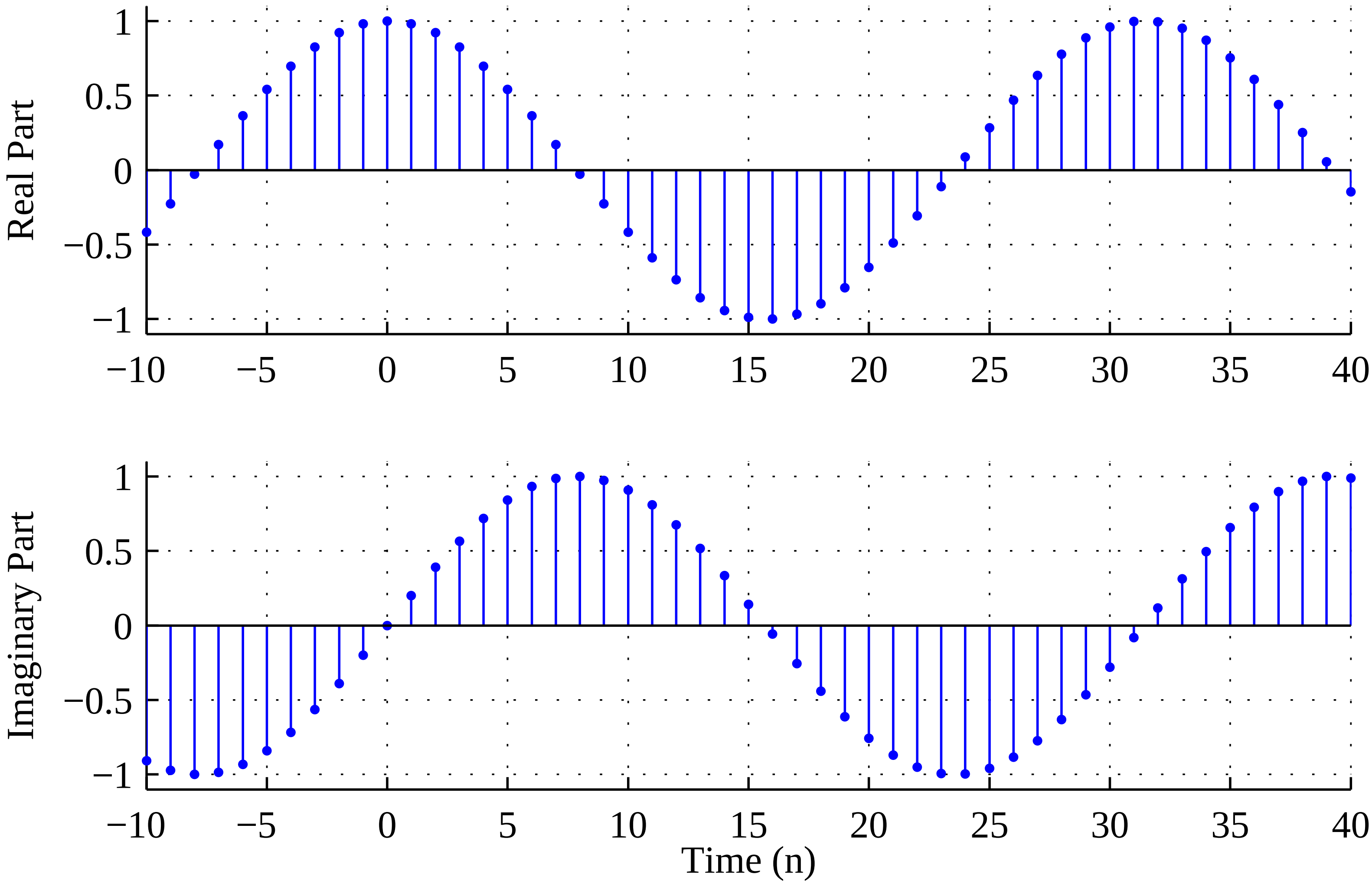
Discrete-Time Signal Intuition

$$x[n] = \sum_k a_k e^{j\Omega_k n} \qquad \rightarrow \qquad y[n] = \sum_k a_k H\left(e^{j\Omega_k}\right) e^{j\Omega_k n}$$

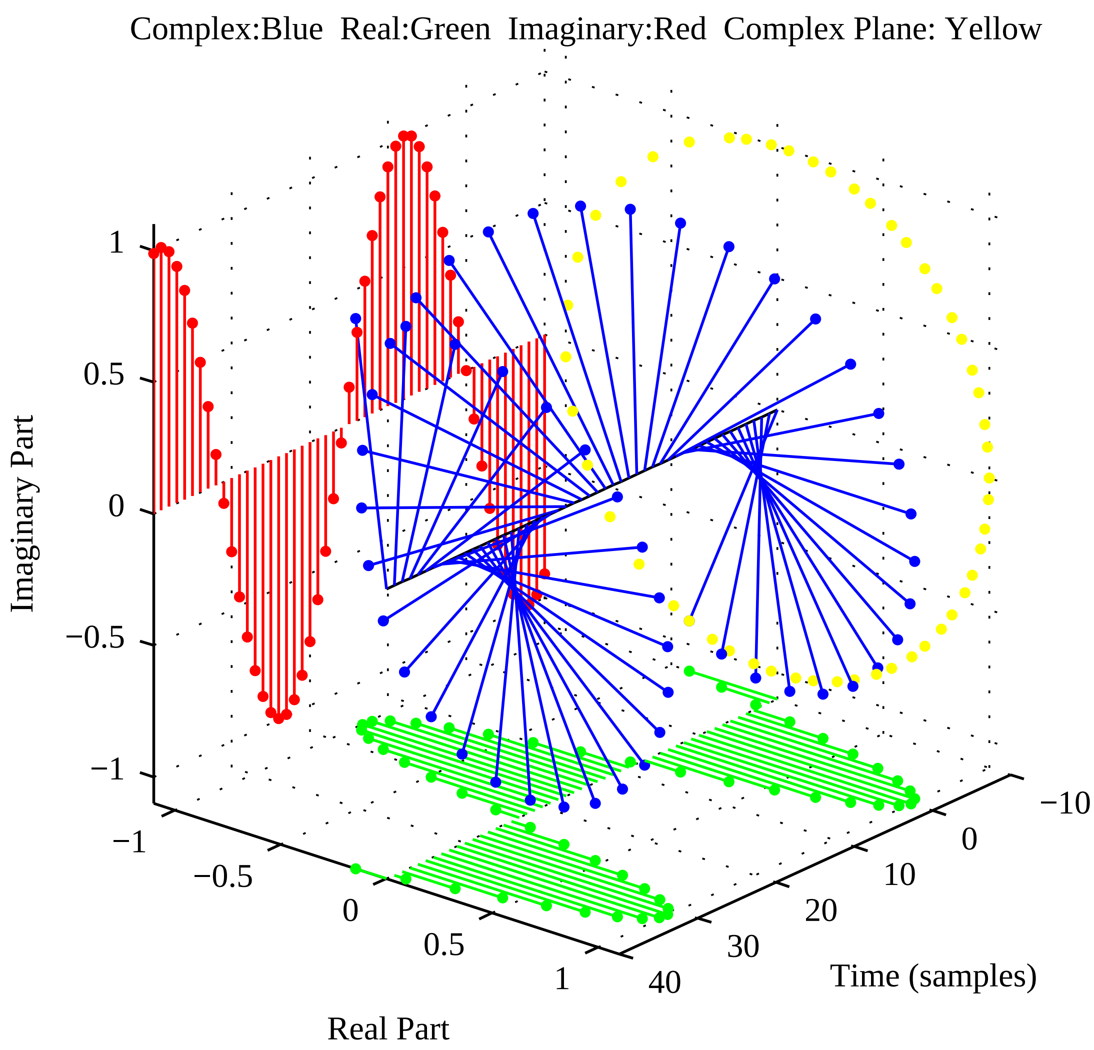
- It is similarly worthwhile to understand discrete-time complex exponentials as thoroughly as possible

$$x[n] \quad = \quad z^n\big|_{|z|=1} = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$$

Example 3: $x[n] = e^{j0.2n}$



Example 3: $x[n] = e^{j0.2n}$



Example 3: MATLAB Code

```
n  = -10:40;           % Time index
N  = length(n);
w  = 0.2;
y  = exp(j*w*n);
subplot(2,1,1);
   h = stem(n,real(y));
   set(h(1),'Marker','.');
   box off;
   grid on;
   ylim([-1.1 1.1]);
   ylabel('Real Part');
subplot(2,1,2);
   h = stem(n,imag(y));
   set(h(1),'Marker','.');
   box off;
   grid on;
   ylim([-1.1 1.1]);
   xlabel('Time (n)');
   ylabel('Imaginary Part');
```

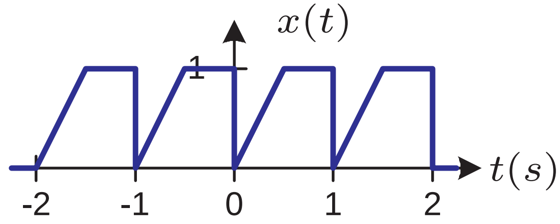
Example 3: MATLAB Code Continued

```
h = plot3(n,zeros(1,N),zeros(1,N),'k');
hold on;
   h = plot3(ones(2,1)*n,[zeros(1,N);imag(y)],[zeros(1,N);real(y)],'b');
   h = plot3(n,imag(y),real(y),'b.');
   h = plot3(ones(2,1)*n,1.1*ones(2,N),[zeros(1,N);real(y)],'r');
   h = plot3(n,1.1*ones(1,N),real(y),'r.');
   h = plot3(ones(2,1)*n,[zeros(1,N);imag(y)],-1.1*ones(2,N),'g');
   h = plot3(n,imag(y),-1.1*ones(size(n)),'g.');
   hold off;
grid on;
ylabel('Imaginary Part');
zlabel('Real Part');
title('Complex:Blue  Real:Red  Imaginary:Green');
axis([min(n) max(n) -1.1 1.1 -1.1 1.1]);
view(27.5,22);
```

Signal Classes

- There are four classes of signals
- There is a different transform for each class
- Periodic
  - Continuous-time: CT Fourier series
  - Discrete-time: DT Fourier series
- Nonperiodic
  - Continuous-time: CT Fourier transform
  - Discrete-time: DT Fourier transform
- Fourier series can only be applied to periodic signals
- The Fourier transforms are best applied to signals that are nonperiodic

Definition of Continuous-Time Periodic Signals



- A signal $x(t)$ is **periodic** if there exists a $T > 0$ such that

$$x(t+T) = x(t) \text{ for all } t$$

- **Fundamental period**: the minimum value of T for which the above holds. Often denoted as T_0.
- **Fundamental frequency**:
  - $f_0 \triangleq \frac{1}{T_0} = \frac{\omega_0}{2\pi}$ Hz
  - $\omega_0 \triangleq \frac{2\pi}{T_0} = 2\pi f_0$ rad/s
- Will frequently drop the 0 subscript to simplify notation

Continuous-Time Exponential Harmonics

- Last term we discussed harmonically-related periodic signals
- For complex sinusoids

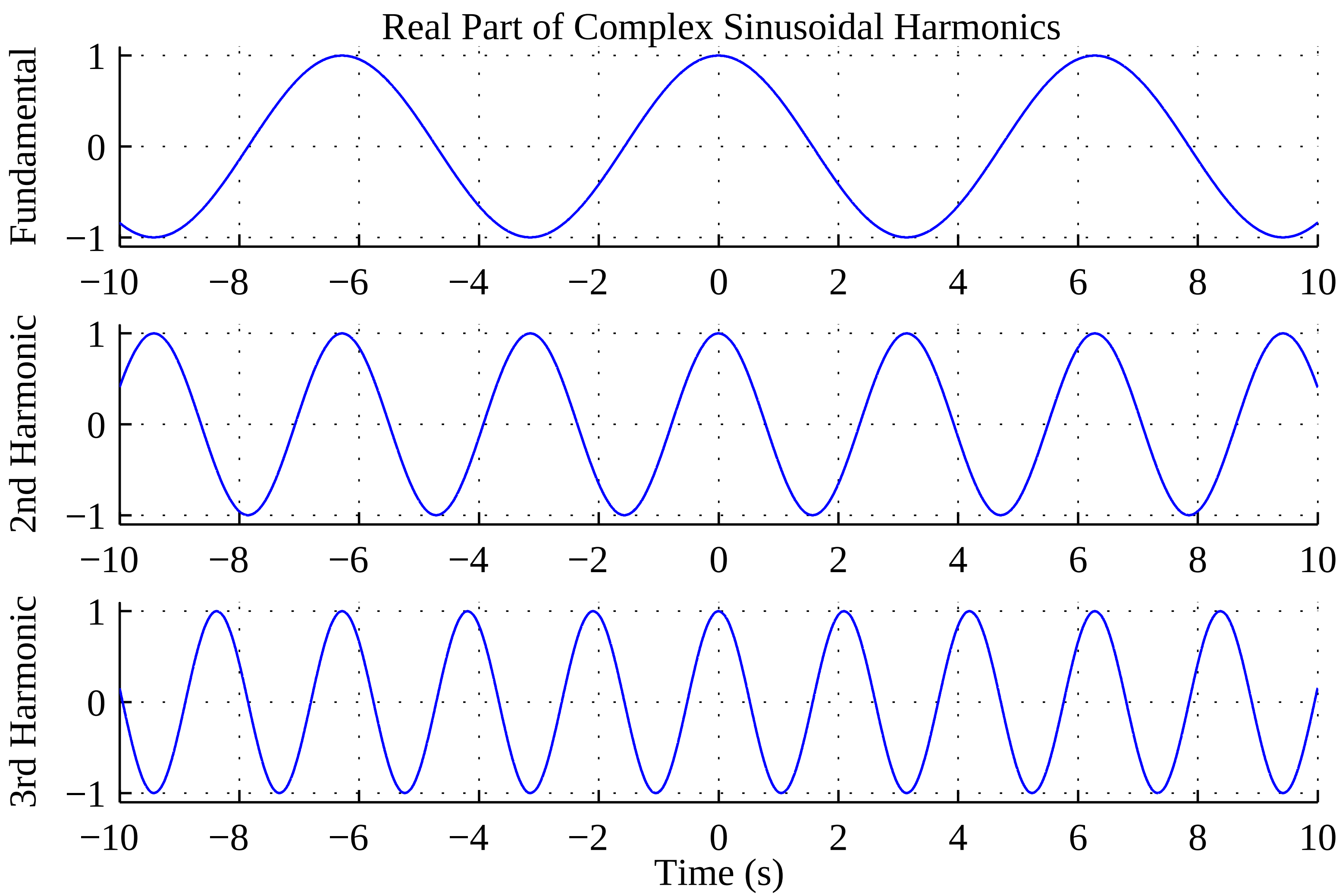
$$\mathrm{e}^{jk\omega t} \quad k = 0, \pm 1, \pm 2, \ldots$$

- $\mathrm{e}^{j\omega t}$ is called the **fundamental component**
- $\mathrm{e}^{j2\omega t}$ is called the **2nd harmonic component**
- $\mathrm{e}^{j3\omega t}$ is called the **3rd harmonic component**
- Key idea: A sum of harmonically related exponentials still has the fundamental frequency ω

$$x(t) = \sum_k a_k \mathrm{e}^{jk\omega t}$$

- Each term in the sum has one or more complete cycles every T seconds

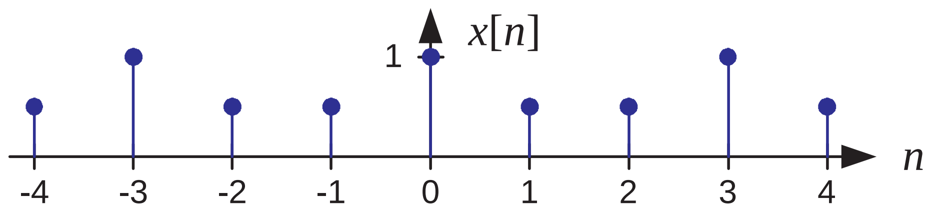
Example 4: $x(t) = e^{jt}$



Example 4: MATLAB Code

```
w  = j*1;
fs = 500;                 % Sample rate (Hz)
t  = -10:1/fs:10;         % Time index (s)
N  = length(t);
subplot(3,1,1);
   h = plot(t,real(exp(1*w*t)));
   box off;
   grid on;
   ylim([-1.1 1.1]);
   ylabel('Fundamental');
subplot(3,1,2);
   h = plot(t,real(exp(2*w*t)));
   box off;
   grid on;
   ylim([-1.1 1.1]);
   ylabel('2nd Harmonic');
subplot(3,1,3);
   h = plot(t,real(exp(3*w*t)));
   box off;
   grid on;
   ylim([-1.1 1.1]);
   ylabel('3rd Harmonic');
   xlabel('Time (s)');
```

Definition of Discrete-Time Periodic Signals



- A signal $x[n]$ is **periodic** if there exists an *integer* $N > 0$ such that

$$x[n+N] = x[n] \text{ for all } n$$

- The **fundamental period** N_0 is the minimum value of N for which the above holds
- The **fundamental frequency** is
  - $f_0 \triangleq \frac{1}{N_0} = \frac{\omega_0}{2\pi}$ cycles/sample
  - $\omega_0 \triangleq \frac{2\pi}{N_0} = 2\pi f_0$ rad/sample
- Will frequency drop the 0 subscript to simplify notation

Example 5: Discrete-Time Periodic Signals

Determine which of the following discrete-time signals are periodic: $\sin(n)$, $\cos(5n)$, $\cos(2\pi n)$, $\cos(2\pi \frac{17}{39} n + 1.238424)$. If the signal is periodic, find the fundamental period. If not, explain why the signal is not periodic.

Example 5: Workspace

Periodic Signals: Sinusoids

By Euler's identity, sinusoids can be expressed as a sum of complex sinusoids

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ \cos(\omega t) &= \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) &= \tfrac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{aligned}$$

- Any sum of sinusoids can be expressed as a sum of complex exponentials
- Any sum of complex sinusoids can be expressed as a sum of sinusoids
- Conceptually, this is a good way to think of complex sinusoids

Periodic Signals as Sums of Sinusoids

$$\hat{x}(t) = \sum_k a_k \mathrm{e}^{jk\omega t} \qquad \rightarrow \qquad y(t) = \sum_k a_k H(jk\omega) \mathrm{e}^{jk\omega t}$$

$$\hat{x}[n] = \sum_k a_k \mathrm{e}^{jk\Omega n} \qquad \rightarrow \qquad y[n] = \sum_k a_k H(\mathrm{e}^{jk\Omega}) \mathrm{e}^{jk\Omega n}$$

- Suppose we wish to represent periodic signals as sums of complex sinusoids to easily calculate and understand the outputs of LTI systems
- Note that *if the input signal to an LTI system is periodic, the output signal will also be periodic with the same fundamental period*
- If the signals are periodic, the complex sinusoids must be harmonically related (all must repeat during the fundamental period)
- We'll consider discrete-time (DT) periodic signals first

Discrete-Time Exponential Harmonics

- The set of all discrete-time complex sinusoidal signals that are periodic with period N can be expressed as

$$\mathrm{e}^{jk\Omega n} = \mathrm{e}^{jk\frac{2\pi}{N}n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- $\mathrm{e}^{j\Omega n}$ is called the **fundamental component**
- $\mathrm{e}^{j2\Omega n}$ is called the **2nd harmonic component**
- $\mathrm{e}^{jk\Omega n}$ is called the k**th harmonic component**

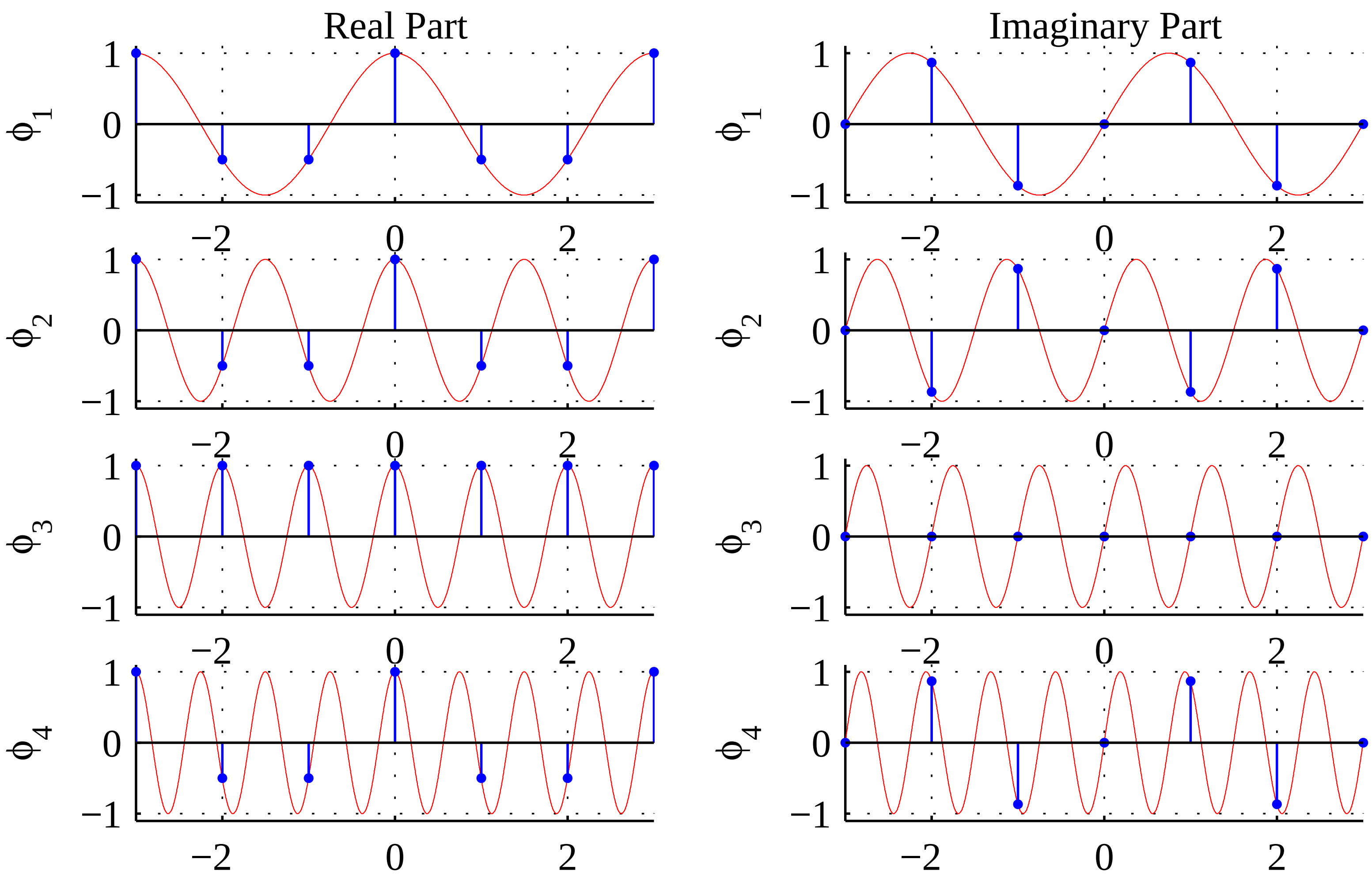
Discrete-Time Exponential Harmonics Redundancy

Unlike continuous-time exponentials, there are only N distinct harmonics

$$
\begin{aligned}
e^{jk\Omega n} &= e^{jk\frac{2\pi}{N}n} \\
e^{j(k+\ell N)\Omega n} &= e^{j(k+\ell N)\frac{2\pi}{N}n} \\
&= e^{jk\frac{2\pi}{N}n + j\ell 2\pi n} \\
&= e^{jk\frac{2\pi}{N}n} e^{j\ell 2\pi n} \\
&= e^{jk\frac{2\pi}{N}n} \\
e^{j(k+\ell N)\Omega n} &= e^{jk\Omega n}
\end{aligned}
$$

This is an *very important difference* between the DT and CT signals. This will be especially important when we discuss sampling.

Example 6: Discrete-Time Harmonics ($N = 3$)



Example 6: MATLAB Code

```
N  = 3;
w  = 2*pi/N;
fs = 500;               % Real-Time Sample rate (Hz)
t  = -3:1/fs:3;         % Time index (s)
n  = -3:1:3;            % Sample index
for cnt = 1:4,
   subplot(4,2,cnt*2-1);
      h = plot(t,real(exp(j*cnt*w*t)),'r');
      set(h,'LineWidth',0.1);
      hold on;
            h = stem(n,real(exp(j*cnt*w*n)),'.');
            hold off;
      box off;
      grid on;
      xlim([min(t) max(t)]);
      ylim([-1.1 1.1]);
      ylabel(sprintf('\\phi_%d',cnt));
```

Example 6: MATLAB Code Continued

```
subplot(4,2,cnt*2);
   h = plot(t,imag(exp(j*cnt*w*t)),'r');
   set(h,'LineWidth',0.1);
   hold on;
         h = stem(n,imag(exp(j*cnt*w*n)),'.');
         hold off;
   box off;
   grid on;
   xlim([min(t) max(t)]);
   ylim([-1.1 1.1]);
   ylabel(sprintf('\\phi_%d',cnt));
end;
```

Summary

- We are interested in complex sinusoids because they are eigenfunctions of LTI systems
  - This makes it easy to compute the output of LTI systems
  - This gives us insight into what LTI systems do
- There are many similarities between CT and DT signals
- There are also some critical differences
  - DT signals are only periodic if $x[n+N] = x[n]$ for some *integer* N
  - There are only N *distinct* DT complex sinusoidal harmonics that have a period N
- This last idea is crucial to this class