1. DT Fourier Transform Properties.

a. For which of the following signals does the DTFT converge:

\[ x[n] = e^{-j0.1\pi n} \]
\[ x[n] = e^{j0.1\pi n} \]
\[ x[n] = 0 \]
\[ x[n] = u[n] - u[n-3] - u[n+3] \]
\[ x[n] = u[3] \]

The general rule is that the DTFT only converges for energy signals, which are those for which

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \]

Of the signals above, this is only satisfied for the following signals:

\[ x[n] = e^{(-0.1\pi)n}u[n] \]
\[ x[n] = e^{0.1\pi n}u[-n] \]
\[ x[n] = u[n] \]
\[ x[n] = u[n-3] - u[n+3] \]
\[ x[n] = u[n+3] - u[n-3] \]

b. What does it mean for the DTFT to converge?

It means that \(|X(e^{j\Omega})| < \infty\) for all \(\Omega\).

c. If a signal is real-valued, what does this tell you about the DTFT?

That the DTFT has complex conjugate symmetry: \(X(e^{-j\Omega}) = X^*(e^{j\Omega})\).

d. If a signal is real and has even or odd symmetry, what else does this tell you about the DTFT?

If the signal is real and even, the DTFT is real (the imaginary part is zero) and has even symmetry, \(X(e^{-j\Omega}) = X(e^{j\Omega})\). If the signal is real and odd, the DTFT is imaginary (the real part is zero) and has odd symmetry, \(X(e^{-j\Omega}) = -X(e^{j\Omega})\).
e. Why is the DTFT a periodic function of frequency? Essentially because discrete time complex sinusoids are not distinct: $e^{j\Omega n} = e^{j(\Omega+2\pi)n}$. The periodicity of the DTFT then follows directly

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\Omega+2\pi)n}$$

$$= X(e^{j(\Omega+2\pi)})$$

f. Classify each of the following signals as high, mid, or low-frequency signals:

- $x[n] = \cos(\pi n/12)$
- $x[n] = \cos(\pi n6/12)$
- $x[n] = \cos(\pi n11/12)$
- $x[n] = \cos(\pi n25/12)$
- $x[n] = \cos(\pi n49/12)$
- $x[n] = \cos(\pi n59/12)$

High, mid, and low frequency signals are those with most of their energy or power concentrated near $\pi$, $\pi/2$, or 0 radians per sample.

- $\cos(\pi n1/12)$ Low
- $\cos(\pi n6/12)$ Mid
- $\cos(\pi n11/12)$ High
- $\cos(\pi n25/12) = \cos(\pi n1/12)$ Low
- $\cos(\pi n49/12) = \cos(\pi n1/12)$ Low
- $\cos(\pi n59/12) = \cos(\pi n11/12)$ High

2. DT Fourier Transform. Consider the following discrete-time signal

$$x[n] = \begin{cases} 
-1 & n = -3, -1, 1, 3 \\
1 & n = -2, 0, 2 \\
0 & \text{Otherwise}
\end{cases}$$

a. Is this an energy signal or power signal? 

Clearly an energy signal because it has finite duration.

b. Does the signal have even or odd symmetry? 

Yes, it is even.

c. Does this signal primarily have mid, low, or high-frequency components? 

The signal oscillates between extreme values every sample so I would expect most of the energy to be concentrated at high frequencies (near $\pi \pm 2\pi \ell$ for any integer $\ell$).
d. What is the discrete-time Fourier transform of this signal?

\[
X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}
\]
\[
= -e^{j3\Omega} + e^{j2\Omega} - e^{j\Omega} + 1 - e^{-j\Omega} + e^{-j2\Omega} - e^{-j3\Omega}
\]
\[
= 1 - 2 \cos(3\Omega) + 2 \cos(2\Omega) - 2 \cos(\Omega)
\]

e. Plot the signal.

![Signal Plot](image)

f. Plot the real and imaginary parts of the Fourier transform over a range of \( \Omega = -5\pi \) to \( \Omega = 5\pi \).

![Real Part Plot](image)

![Imaginary Part Plot](image)

g. Plot the magnitude and phase of the DTFT \( X(e^{j\Omega}) \).
h. What type of symmetry do the coefficients have?

*Complex conjugate symmetry:* $X[-k] = X^*[k]$. *Since the coefficients are real in this case, they also have even symmetry:* $X[-k] = X[k]$.

i. What is the fundamental period of the DTFT?

$2\pi$, as it always is for the DTFT.

```matlab
function [] = DTFTExample

n = -10:10; % Time indices
Omega = linspace(-5*pi,5*pi,500); % Frequency indices
x = (abs(n)==2 | n==0) - (abs(n)==3 | abs(n)==1);
X = 1-2*cos(3*Omega)+2*cos(2*Omega)-2*cos(Omega);

%====================================================================
% Plot of the Signal
%====================================================================
figure;
FigureSet(1,5);
h = stem(n,x,'b');
set(h,'Marker','.');
xlim([n(1)-0.5 n(end)+0.5]);
ylim([-1.05 1.05]);
xlabel('Time Index (n)');
box off;
AxisLines;
AxisSet(8);
print('DTFTExample-Signal','-depsc');

%====================================================================
% Plot of the Fourier Series Coefficients (Rectangular form)
%====================================================================
figure;
FigureSet(1,5);
subplot(2,1,1);
h = plot(Omega,real(X),'r');
xlim([Omega(1) Omega(end)]);
ylim([1.05*max(abs(X))*[-1 1]]);
h = get(gca, 'YLabel');
set(h,'Interpreter','LaTeX');
```

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ylabel('Real $X(e^{j\Omega})$');
box off;
AxisLines;
subplot(2,1,2);
    h = plot(Omega,imag(X),'r');
xlim([Omega(1) Omega(end)]);
ylim(1.05*max(abs(X))*[-1 1]);
h = get(gca,'YLabel');
set(h,'Interpreter','LaTeX');
h = get(gca,'XLabel');
set(h,'Interpreter','LaTeX');
ylabel('Imaginary $X(e^{j\Omega})$');
xlabel('Frequency ($\Omega$)');
box off;
AxisLines;
print('DTFTExample-Rectangular','-depsc');

%====================================================================

% Plot of the Fourier Series Coefficients (Rectangular form)
%====================================================================
figure;
FigureSet(1,5);
subplot(2,1,1);
    h = plot(Omega,abs(X),'r');
xlim([Omega(1) Omega(end)]);
ylim([0 1.05*max(abs(X))]);
h = get(gca, 'YLabel');
set(h,'Interpreter','LaTeX');
ylabel('$|X(e^{j\Omega})|$');
box off;
AxisLines;
subplot(2,1,2);
    h = stairs(Omega,(180/pi)*angle(X),'r');
xlim([Omega(1) Omega(end)]);
ylim([190*[-1 1]]);
h = get(gca, 'YLabel');
set(h,'Interpreter','LaTeX');
h = get(gca,'XLabel');
set(h,'Interpreter','LaTeX');
ylabel('$\angle X(e^{j\Omega})$ (degrees)');
xlabel('Frequency ($\Omega$)');
box off;
AxisLines;
AxisSet(8);
print('DTFTExample-Polar','-depsc');