Two-Port Networks

- Definitions
- Impedance Parameters
- Admittance Parameters
- Hybrid Parameters
- Transmission Parameters
- Cascaded Two-Port Networks
- Examples
- Applications
A pair of terminals at which a signal (voltage or current) may enter or leave is called a **port**.

A network having only one such pair of terminals is called a **one-port network**.

No connections may be made to any other nodes internal to the network.

By KCL, we therefore have $i_1 = i'_1$.

We discussed in ECE 221 how one-port networks may be modeled by their Thévenin or Norton equivalents.
Two-port networks are used to describe the relationship between a pair of terminals.

The analysis methods we will discuss require the following conditions be met:

1. Linearity
2. No independent sources inside the network
3. No stored energy inside the network (zero initial conditions)
4. \( i_1 = i'_1 \) and \( i_2 = i'_2 \)
Two-Port Networks: Defining Equations

- If the network contains dependent sources, one or more of the equivalent resistors may be negative
- Generally, the network is analyzed in the $s$ domain
- Each two-port has exactly two governing equations that can be written in terms of any pair of network variables
- Like Thévenin and Norton equivalents of one-ports, once we know a set of governing equations we no longer need to know what is inside the box
Impedance Parameters

\[ \begin{align*}
V_1 &= z_{11}I_1 + z_{12}I_2 \\
V_2 &= z_{21}I_1 + z_{22}I_2 \\
\begin{bmatrix} V_1 \\
V_2 \end{bmatrix} &= \begin{bmatrix} z_{11} & z_{12} \\
z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\
I_2 \end{bmatrix}
\end{align*} \]

• Suppose the currents and voltages can be measured
• Alternatively, if the circuit in the box is known, \( V_1 \) and \( V_2 \) can be calculated based on circuit analysis
• Relationship can be written in terms of the **impedance** parameters
• We can also calculate the impedance parameters after making two sets of measurements
Impedance Parameter Measurements

If the right port is an open circuit \((I_2 = 0)\), then we can easily solve for two of the impedance parameters:

\[
\begin{align*}
V_1 &= z_{11}I_1 + z_{12}I_2 \\
V_2 &= z_{21}I_1 + z_{22}I_2
\end{align*}
\]

\[
\begin{align*}
\left. z_{11} = \frac{V_1}{I_1} \right|_{I_2=0} \\
\left. z_{21} = \frac{V_2}{I_1} \right|_{I_2=0}
\end{align*}
\]
Impedance Parameter Measurements Continued

If the left port is an open circuit \((I_1 = 0)\), then we can easily solve for the other two impedance parameters:

\[
\begin{align*}
V_1 &= z_{11} I_1 + z_{12} I_2 \\
V_2 &= z_{21} I_1 + z_{22} I_2 \\
\end{align*}
\]

\[z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0} \quad \quad \quad \quad z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0}\]
Impedance Parameter Measurements Summarized

\[ z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=0} \]
\[ z_{21} = \frac{V_2}{I_1} \bigg|_{I_2=0} \]
\[ z_{12} = \frac{V_1}{I_2} \bigg|_{I_1=0} \]
\[ z_{22} = \frac{V_2}{I_2} \bigg|_{I_1=0} \]
Impedance Parameter Equivalent

\[ I_1(s) \rightarrow + \quad + \quad z_{11} \quad + \quad z_{12}I_2 \quad - \quad + \quad z_{21}I_1 \quad - \quad + \quad I_2(s) \rightarrow \]

\[ V_1(s) \quad - \quad - \quad V_2(s) \quad - \quad - \]

\[ V_1 = z_{11}I_1 + z_{12}I_2 \]
\[ V_2 = z_{21}I_1 + z_{22}I_2 \]

- Once we know what the impedance parameters are, we can model the behavior of the two-port with an equivalent circuit.
- Notice the similarity to Thévenin and Norton equivalents.
Example 1: Impedance Parameters

Find the $z$ parameters of the circuit.
Example 1: Workspace
Example 2: Parameter Conversion

\[ V_1(s) = z_{11}I_1 + z_{12}I_2 \]
\[ V_2(s) = z_{21}I_1 + z_{22}I_2 \]

In general, the two defining equations can be written in terms of any pair of variables. For example, rewrite the defining equations in terms of the voltages \( V_1 \) and \( V_2 \).
Example 2: Workspace
**Impedance & Admittance Parameters**

\[ I_1(s) \rightarrow \text{Two-Port Network} \rightarrow I_2(s) \]

\begin{align*}
V_1(s) & = z_{11} I_1 + z_{12} I_2 \\
V_2(s) & = z_{21} I_1 + z_{22} I_2
\end{align*}

**Impedance Parameters**

\[
\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}
\]

**Admittance Parameters**

\[
\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}
\]
Hybrid Parameters

\[ I_1(s) \rightarrow \text{Two-Port Network} \leftarrow I_2(s) \]

\[ V_1(s) \]
\[ \quad + \]
\[ V_2(s) \]
\[ \quad + \]

\[ + \]
\[ V_1(s) \]
\[ \quad - \]
\[ - \]

Hybrid Parameters

\[ V_1 = h_{11} I_1 + h_{12} V_2 \]
\[ I_2 = h_{21} I_1 + h_{22} V_2 \]

\[
\begin{bmatrix}
V_1 \\
I_2
\end{bmatrix}
= \begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
V_2
\end{bmatrix}
\]

Inverse Hybrid Parameters

\[ I_1 = g_{11} V_1 + g_{12} I_2 \]
\[ V_2 = g_{21} V_1 + g_{22} I_2 \]

\[
\begin{bmatrix}
I_1 \\
V_2
\end{bmatrix}
= \begin{bmatrix}
g_{11} & g_{12} \\
g_{21} & g_{22}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
I_2
\end{bmatrix}
\]
Transmission Parameters

\[ V_1(s) = a_{11}V_2(s) - a_{12}I_2(s) \]
\[ I_1(s) = a_{21}V_2(s) - a_{22}I_2(s) \]

\[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = A \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \]

Inverse Transmission Parameters

\[ V_2(s) = b_{11}V_1(s) - b_{12}I_1(s) \]
\[ I_2(s) = b_{21}V_1(s) - b_{22}I_1(s) \]

\[ \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix} = B \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \]
Transmission Parameter Conversion

- Altogether there are 6 sets of parameters
- Each set completely describes the two-port network
- Any set of parameters can be converted to any other set
- We have seen one example of a conversion
- A complete table of conversions is listed in the text (Pg. 933)
- You should have a copy of this in your notes for the final
Example 3: Two-Port Measurements

The following measurements were taken from a two-port network. Find the transmission parameters.

Port 2 Open

\[ V_1 = 150 \cos(4000t) \text{ V applied} \]
\[ I_1 = 25 \cos(4000t - 45^\circ) \text{ A measured} \]
\[ V_2 = 1000 \cos(4000t + 15^\circ) \text{ V measured} \]

Port 2 Shorted

\[ V_1 = 30 \cos(4000t) \text{ V applied} \]
\[ I_1 = 1.5 \cos(4000t + 30^\circ) \text{ A measured} \]
\[ I_2 = 0.25 \cos(4000t + 150^\circ) \text{ A measured} \]
Example 3: Workspace
Example 4: Two-Port Analysis

Find the hybrid parameters for the circuit shown above.
Example 4: Workspace

\[
\begin{align*}
\begin{array}{c}
\text{\(v_1\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array} & \quad \begin{array}{c}
\text{\(v_3\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array} & \quad \begin{array}{c}
\text{\(v_2\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\(i_1\)} & \quad \begin{array}{c}
\text{\(40 \Omega\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array} & \quad \begin{array}{c}
\text{\(800 \Omega\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array} & \quad \begin{array}{c}
\text{\(160 \Omega\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\(16.2 v_3\)} \\
\text{\(+\)} \\
\text{\(-\)} \\
\text{\(-\)} \\
\end{array}
\end{align*}
\]
Example 4: Workspace Continued
Example 5: Two-Port Measurements

The following measurements were taken from a two-port network. Find the transmission parameters.

<table>
<thead>
<tr>
<th>Port 1 Open</th>
<th>Port 1 Shorted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1 = 1 \text{ mV}$</td>
<td>$I_1 = -0.5 \text{ } \mu\text{A}$</td>
</tr>
<tr>
<td>$V_2 = 10 \text{ V}$</td>
<td>$I_2 = 80 \text{ } \mu\text{A}$</td>
</tr>
<tr>
<td>$I_2 = 200 \text{ } \mu\text{A}$</td>
<td>$V_2 = 5 \text{ V}$</td>
</tr>
</tbody>
</table>

Hint: $\Delta_b = b_{11}b_{22} - b_{12}b_{21}$, $a_{11} = \frac{b_{22}}{\Delta_b}$, $a_{12} = \frac{b_{12}}{\Delta_b}$, $a_{21} = \frac{b_{21}}{\Delta_b}$, and $a_{22} = \frac{b_{11}}{\Delta_b}$. 
Example 5: Workspace
Example 6: Two-Port Analysis

Find an expression for the transfer function, $h_{11}$, $z_{11}$, $g_{12}$, $g_{22}$, $a_{11}$, and $y_{21}$.
Example 6: Workspace

\[ v_1(t) + R_1 i_1 - R_3 v_+(t) + R_4 i_2 = C_1 v_1(t) - C_2 v_2(t) + \]
Example 6: Workspace Continued (1)

$i_1 \rightarrow R_1 \rightarrow R_3 v_+(t) \rightarrow + \leftarrow R_4 \leftarrow i_2$

$v_1(t) \uparrow \rightarrow C_1 \rightarrow R_2 \rightarrow v_-(t) \downarrow$

$C_2$
Example 6: Workspace Continued (2)

\[ v_{1}(t) + R_{1} i_{1} + R_{3} v_{+}(t) - R_{2} v_{-}(t) = C_{1} v_{1}(t) + C_{2} v_{2}(t) \]
Cascaded Two-Port Networks

Two networks are cascaded when the output of one is the input of the other.

Note that $V_{2A} = V_{1B}$ and $-I_{2A} = I_{1B}$.

The transmission parameters take advantage of these properties.
Cascaded Two-Port Networks

\[
\begin{align*}
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_A \begin{bmatrix} V_{2A} \\ -I_{2A} \end{bmatrix} \\
\begin{bmatrix} V_{2B} \\ -I_{2A} \end{bmatrix} &= \begin{bmatrix} V_{1B} \\ I_{1B} \end{bmatrix} \\
\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_A \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \\
\begin{bmatrix} V_{2B} \\ -I_{2A} \end{bmatrix} &= \begin{bmatrix} V_{1B} \\ I_{1B} \end{bmatrix}
\end{align*}
\]
The inverse transmission parameters are also convenient for cascaded networks.

\[
\begin{bmatrix}
  V_2 \\
  I_2
\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\
  b_{21} & b_{22} \end{bmatrix}_A \begin{bmatrix}
  V_{1B} \\
  -I_{1B}
\end{bmatrix} \quad \begin{bmatrix}
  V_{2A} \\
  I_{2A}
\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\
  b_{21} & b_{22} \end{bmatrix}_B \begin{bmatrix}
  V_1 \\
  -I_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
  V_{1B} \\
  -I_{1B}
\end{bmatrix} = \begin{bmatrix}
  V_{2A} \\
  I_{2A}
\end{bmatrix} \quad \begin{bmatrix}
  V_2 \\
  I_2
\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\
  b_{21} & b_{22} \end{bmatrix}_A \begin{bmatrix} b_{11} & b_{12} \\
  b_{21} & b_{22} \end{bmatrix}_B \begin{bmatrix}
  V_1 \\
  -I_1
\end{bmatrix}
\]
Cascaded Systems: Two-Port Networks versus $H(s)$

Two-ports and transfer functions $H(s)$ are closely related

$H(s)$ only relates the input signal to the output signal

Two-ports relate both voltages and currents at each port

You cannot cascade $H(s)$ unless the circuits are active

Two-port networks have no such restriction

Two-ports are used to design passive filters

However, two-ports are more complicated than $H(s)$