Transfer Functions

- Transfer functions defined
- Examples
- System stability
- Pole-Zero Plots
- Sinusoidal steady-state analysis
- Linearity and time invariance defined
- Transfer function synthesis

Transfer Functions

Assume zero initial conditions.

$$\begin{aligned}
\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} &= \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k} \\
\sum_{k=0}^{N} a_k s^k \, Y(s) &= \sum_{k=0}^{M} b_k s^k \, X(s) \\
Y(s) \sum_{k=0}^{N} a_k s^k &= X(s) \sum_{k=0}^{M} b_k s^k \\
Y(s) &= \left(\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \right) X(s) = H(s) X(s)
\end{aligned}$$

Initial Conditions

Assume zero initial conditions.

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

- All voltages and currents are due to independent sources (superposition)
- Energy stored in capacitors and inductors also act like independent sources
- We will now focus a specific class of circuits
  - Only one independent source (input)
  - No energy stored in capacitors or inductors
- Greatly simplifies analysis

Transfer Functions Continued

$$Y(s) = \left(\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \right) X(s) = H(s)X(s)$$

- In the time domain, the relationship can be complicated
- In the s domain, the relationship of $Y(s)$ to $X(s)$ of LTI systems simplifies to a rational function of s
- $H(s)$ is usually a rational ratio of two polynomials
- $H(s)$ is called the **transfer function**
- Specifically, the *transfer function* of an LTI system can be defined as the ratio of $Y(s)$ to $X(s)$
- Usually denoted by $H(s)$, sometimes $G(s)$
- Without loss of generality, usually $a_N \triangleq 1$

Example 1: Transfer Function vs. Impulse Response

Fill in the missing parts to determine how the transfer function of an LTI system $G(s)$ is related to the impulse response $h(t)$

$$x(t) = \delta(t) \qquad X(s) =$$

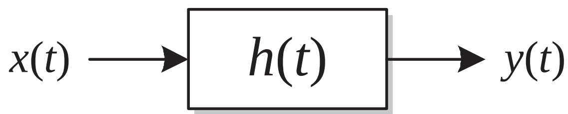
$$y(t) = \qquad Y(s) =$$

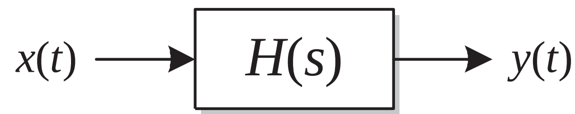
$$\mathcal{L}\{h(t)\} = \qquad \mathcal{L}^{-1}\{G(s)\} =$$

Transfer Functions and the Impulse Response

- Because of their relationship, both $H(s)$ and $h(t)$ completely characterize the LTI system
- If the LTI system is a circuit, once you know either $H(s)$ or $h(t)$, you have sufficient information to calculate the output
- You now have three different approaches to solve for the output of an LTI circuit
  - $y(t) = x(t) * h(t)$
  - Solve for $H(s)$, $X(s)$, and then $y(t) = \mathcal{L}^{-1}\{H(s)\,X(s)\}$
  - Use Laplace transform circuit analysis to solve for the outputs of interest
- All three have limitations, advantages, and disadvantages

Continous-Time Convolution Tradeoffs

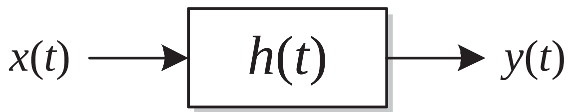


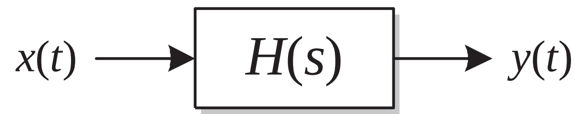


Continuous-time Convolution: $y(t) = x(t) * h(t)$

- Advantages
  - Can find solution for all t, not just $t > 0$
  - Can be approximated using discrete-time convolution
- Disadvantages
  - Cannot account for non-zero initial conditions, requires complete $x(t)$ and $y(t)$
  - Can be difficult to write and solve integrals
  - Can only be used for single-input single-output (SISO) systems that have one independent source

Transfer Function Analysis Tradeoffs

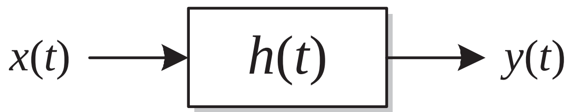


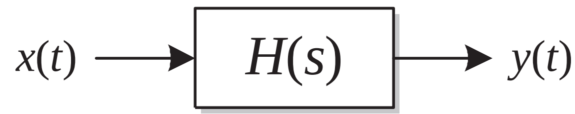


Transfer Function: $y(t) = \mathcal{L}^{-1}\{H(s)\,X(s)\}$

- Advantages
  - Reduces differential equation to an algebra problem
  - Usually the easiest approach
  - Easy to find the output for different input signals
- Disadvantages
  - Can only solve for $y(t)$ for $t > 0$
  - Requires zero initial conditions
  - Can only be used for SISO systems

Laplace Transform Circuit Analysis Tradeoffs

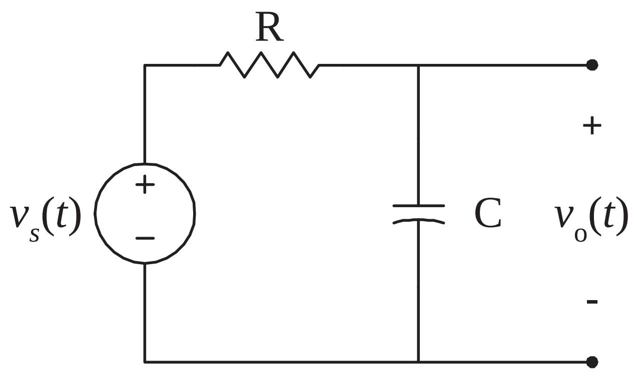




Laplace Transform Circuit Analysis

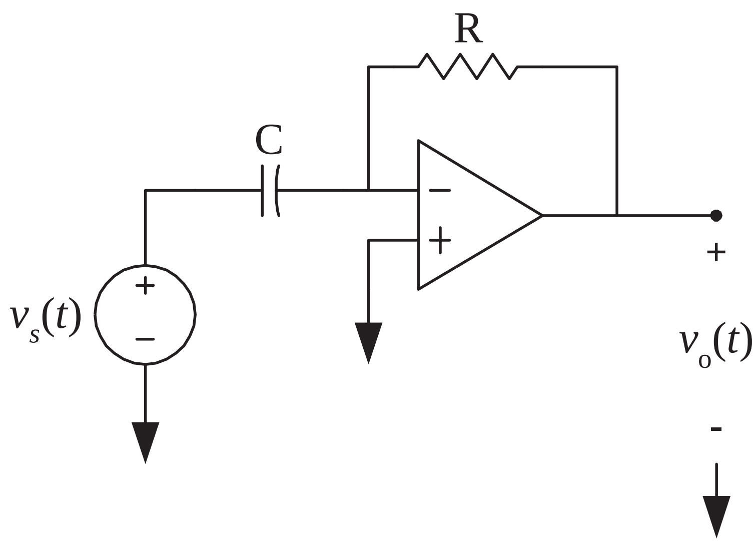
- Advantages
  - Elegant method of handling non-zero initial conditions
  - Can handle multiple sources (multiple inputs) & can solve for multiple outputs (any voltage or current) — MIMO systems
- Disadvantages
  - Can only solve for $y(t)$ for $t > 0$
  - Cannot account for full history, $x(t)$ for $t < 0$. Requires this effect to be captured in the initial conditions
  - Can be tedious
  - Specific to application (circuits), we did not discuss generalization to other types of systems

Example 6: Transfer Functions



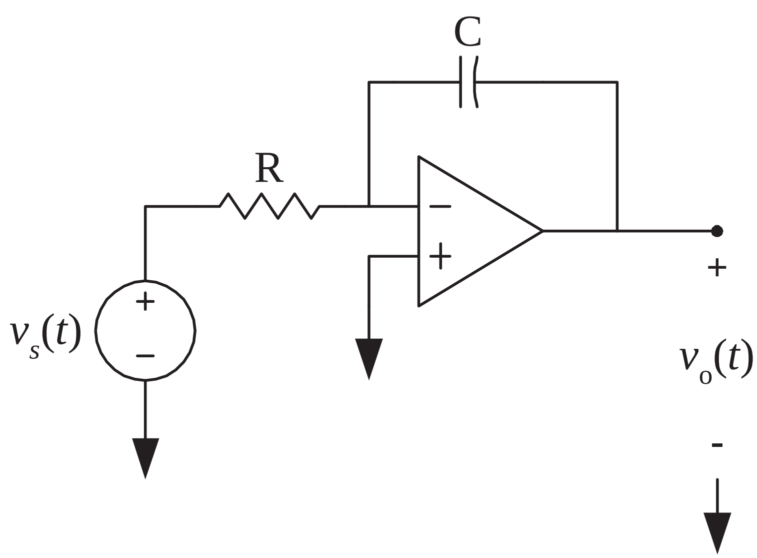
Find the transfer function for the circuit above. The input is the voltage source $v_s(t)$ and the output is labeled $v_o(t)$.

Example 7: Transfer Functions



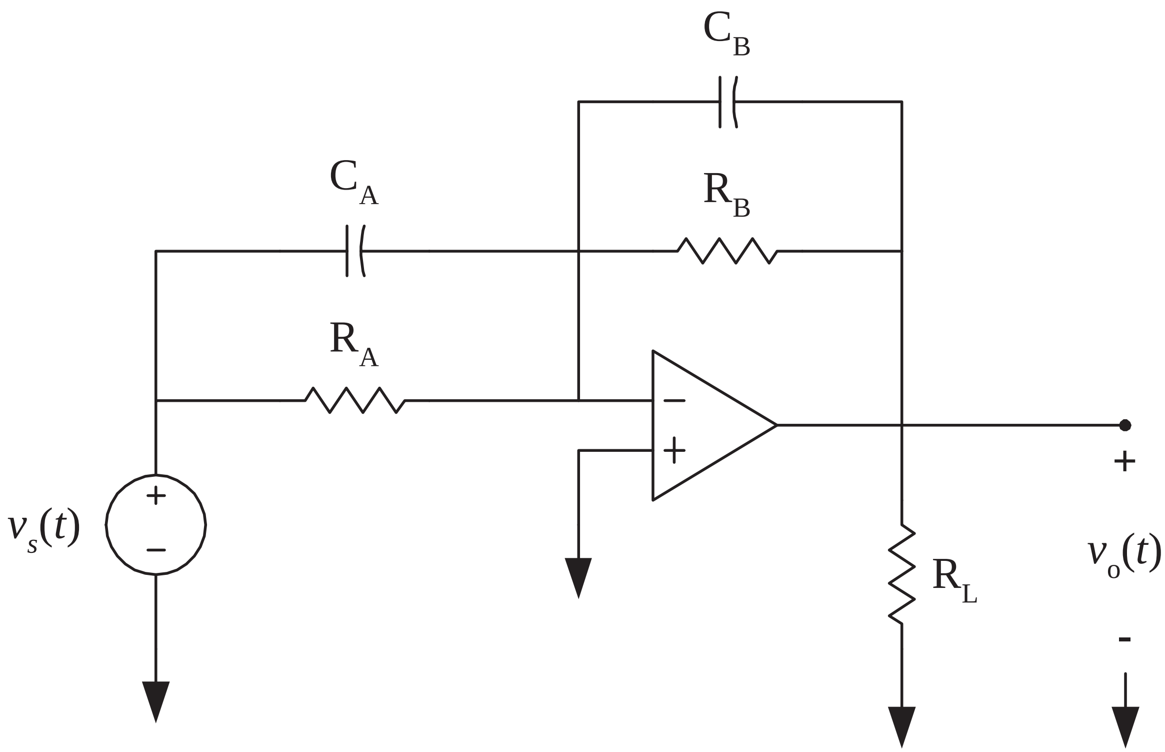
Find the transfer function for the circuit above. Do you recognize this function?

Example 8: Transfer Functions



Find the transfer function for the circuit above. Do you recognize this function?

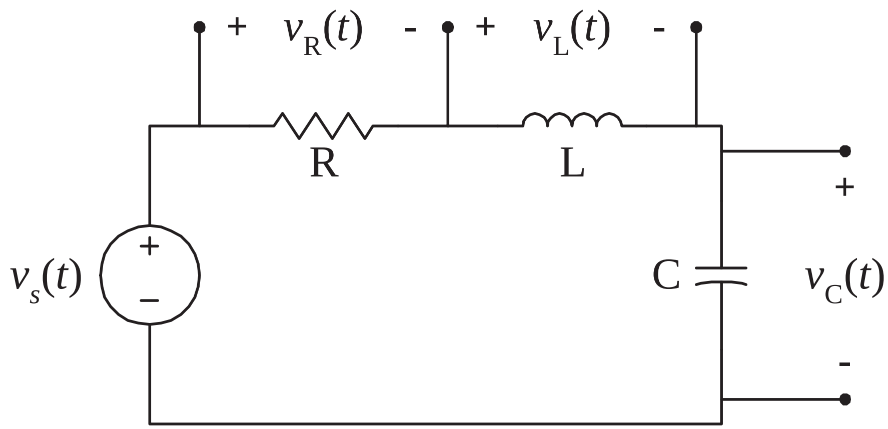
Example 9: Transfer Functions



Find the transfer function for the circuit above.

Example 9: Workspace

Example 10: Transfer Functions



Find the transfer function from the input voltage to an output voltage across each element of the three passive elements in a series RLC circuit.

Example 10: Workspace

Poles and Stability

Assume all of the poles in a transfer function $H(s)$ are unique. Then $H(s)$ can be written as follows using partial fraction expansion:

$$H(s) \quad = \quad \frac{N(s)}{D(s)} = \sum_{\ell=1}^{N} \frac{k_\ell}{s - p_\ell}$$

$$\mathcal{L}^{-1}\left\{H(s)\right\} \quad = \quad h(t) = \sum_{\ell=1}^{N} k_\ell \, \mathrm{e}^{+p_\ell t} u(t)$$

- Note the expansion is in terms of the poles, rather than $-p_\ell$
- If $\int_{-\infty}^{\infty} |h(t)| \, \mathrm{d}t < \infty$, the LTI system is bounded-input bounded-output (BIBO) stable
- That is $|h(t)| < \alpha < \infty$ for all t
- $h(t)$ is bounded if $\mathrm{Re}\{p_\ell\} < 0$ for all ℓ
- The system is BIBO stable if and only if all the poles are in the *left half* of the complex plane

Pole-Zero Plots

$$H(s) = \frac{N(s)}{D(s)}$$

- **Zeros**: roots of $N(s)$
- **Poles**: roots of $D(s)$
- Poles must be in the left half plane for the system to be stable
- As the poles get closer to the boundary, the system becomes less stable
- **Pole-Zero Plot**: plot of the zeros and poles on the complex s plane
- You will use these throughout the junior sequence (ECE 32x)

Example 11: Pole-Zero Plots

Use MATLAB to generate a Pole-Zero plot for a system with the following transfer function

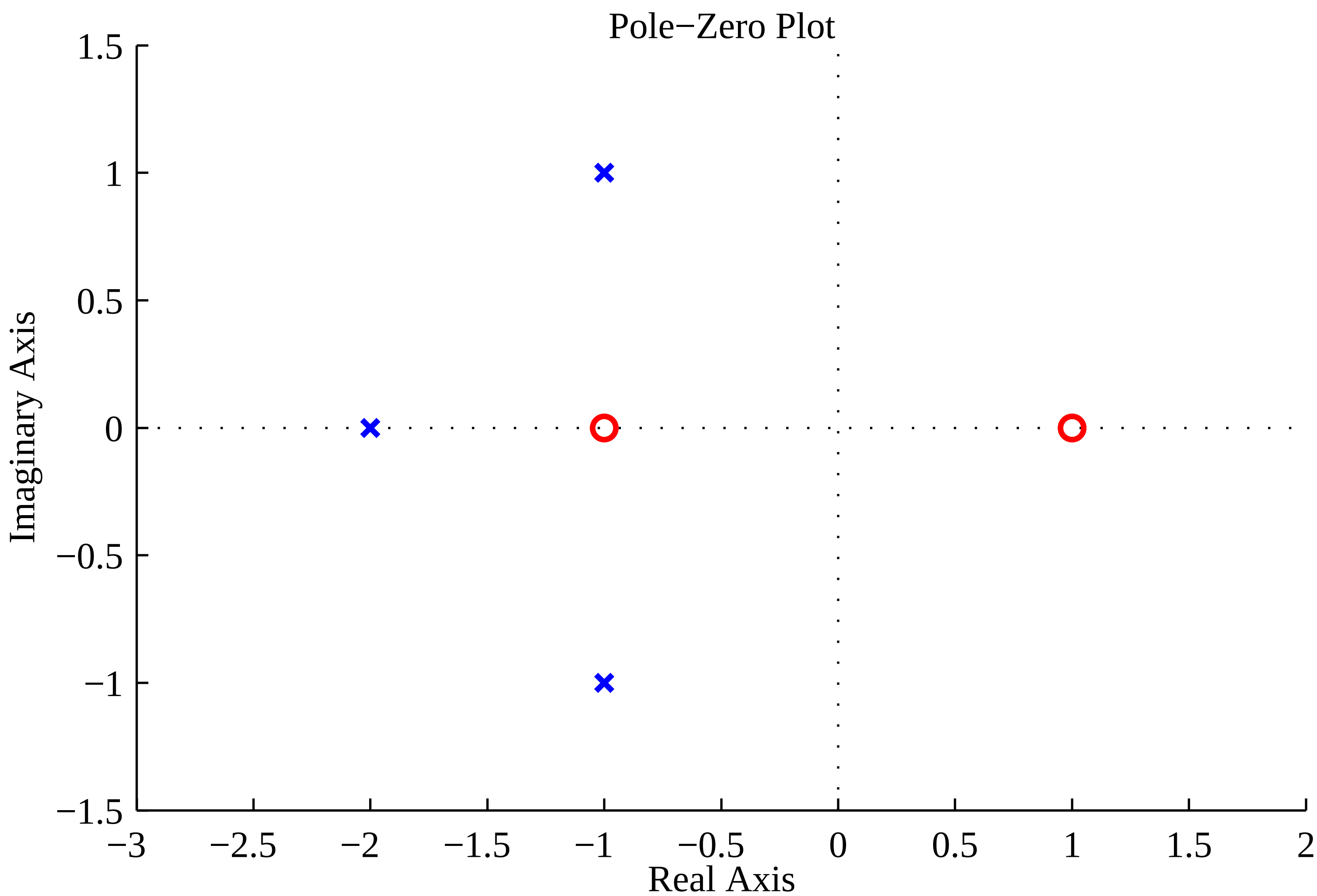
$$H(s) = \frac{s^2 - 1}{s^3 + 4s^2 + 6s + 4}$$

Using the MATLAB, we can quickly find the roots

$$H(s) = \frac{(s+1)(s-1)}{(s+2)(s+1-j)(s+1+j)}$$

Is the system stable? The pole-zero plot, impulse response, and step response are shown on the following slides.

Example 11: Pole-Zero Plot

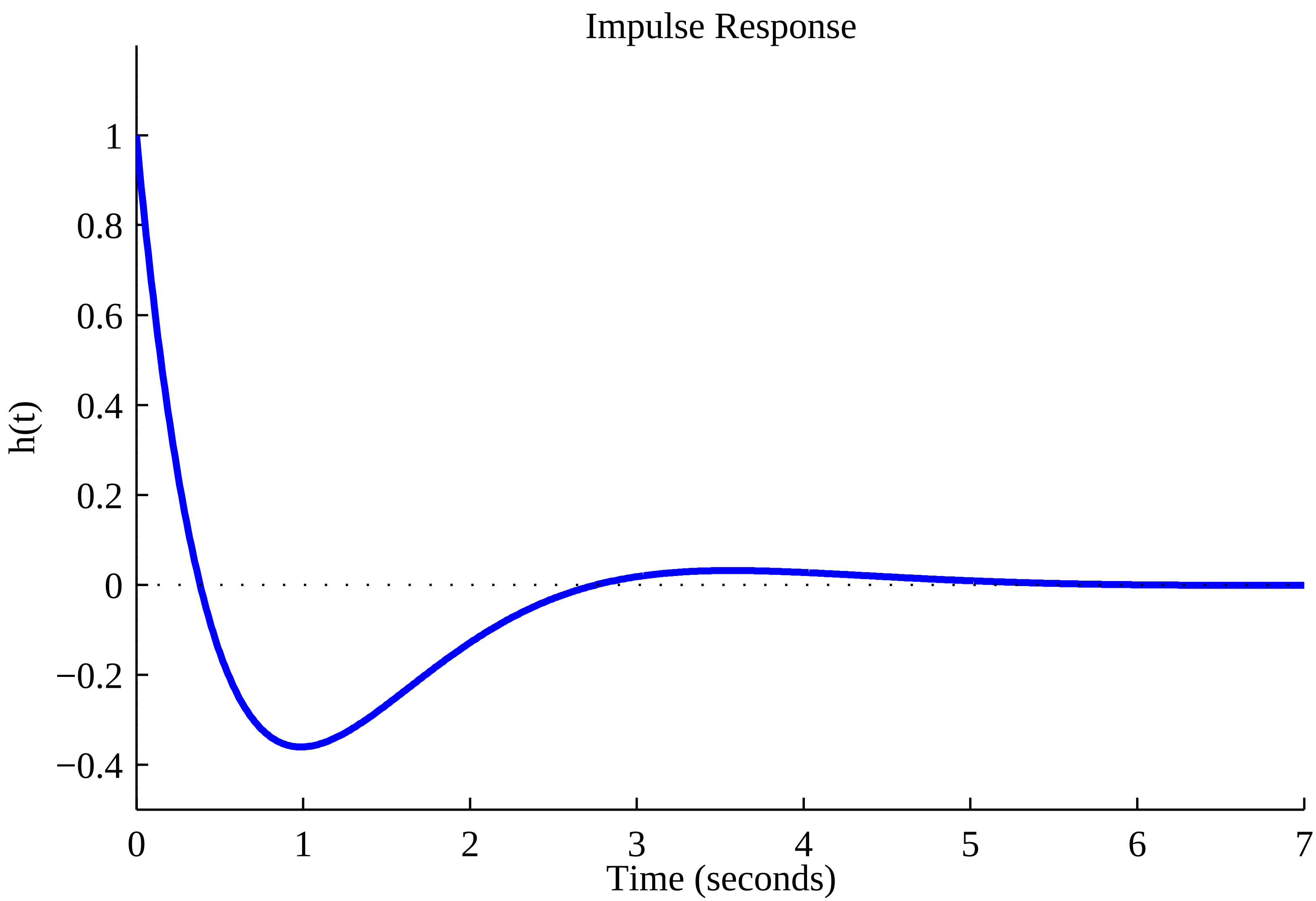


Example 11: MATLAB Code for Pole-Zero Plot

```
sys = tf([1 0 -1],[1 4 6 4]);

figure;
[p,z] = pzmap(sys);
h = plot(real(p),imag(p),'bx',real(z),imag(z),'ro');
set(h,'LineWidth',1.2);
set(h,'MarkerSize',5);
hold on;
   plot([0 0],[-2 2],'k:',[-3 2],[0 0],'k:');
   hold off;
xlabel('Real Axis');
ylabel('Imaginary Axis');
title('Pole-Zero Plot');
axis([-3 2 -1.5 1.5]);
```

Example 11: Impulse Response

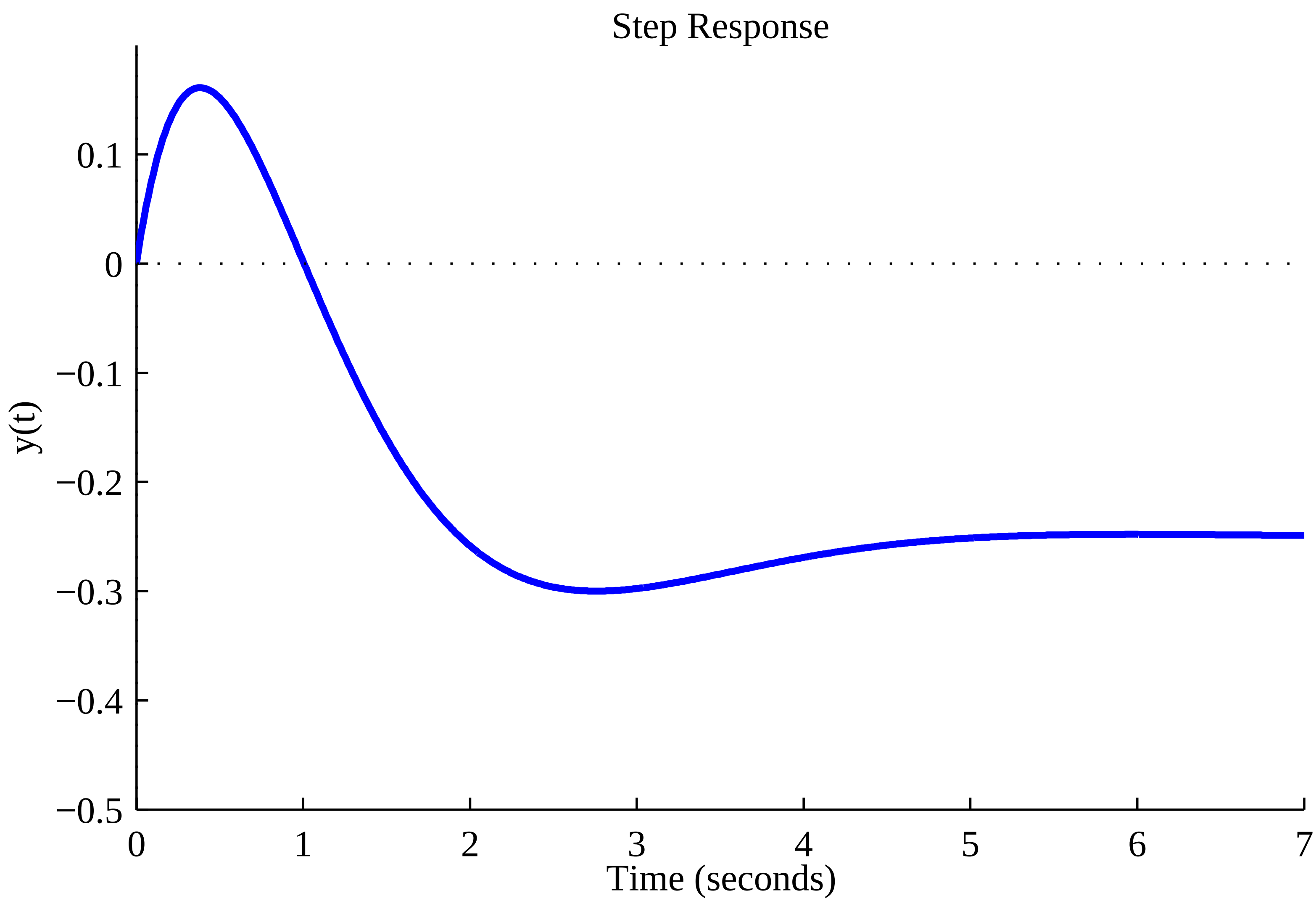


Example 11: MATLAB Code for Impulse Response

```
sys = tf([1 0 -1],[1 4 6 4]);

figure;
t = 0:0.01:7;
[h,t] = impulse(sys,t);
h = plot(t,h);
set(h,'LineWidth',1.5);
hold on;
   plot([0 0],[-2 2],'k:',[0 max(t)],[0 0],'k:');
   hold off;
axis([0 max(t) -0.5 1.2]);
xlabel('Time (seconds)');
ylabel('h(t)');
title('Impulse Response');
```

Example 11: Step Response



Example 11: MATLAB Code for Step Response

```
sys = tf([1 0 -1],[1 4 6 4]);

figure;
t = 0:0.01:7;
[h,t] = step(sys,t);
h = plot(t,h);
set(h,'LineWidth',1.5);
hold on;
   plot([0 0],[-2 2],'k:',[0 max(t)],[0 0],'k:');
   hold off;
axis([0 max(t) -0.5 0.2]);
xlabel('Time (seconds)');
ylabel('y(t)');
title('Step Response');
```

Steady-State Sinusoidal Analysis

Assume a system $H(s)$ is BIBO stable. Consider a sinusoidal input

$$\begin{aligned}
x(t) &= A\cos(\omega t + \phi) \\
&= A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t) \\
\cos(\omega t) &\overset{\mathcal{L}}{\Longleftrightarrow} \frac{s}{s^2+\omega^2} \\
\sin(\omega t) &\overset{\mathcal{L}}{\Longleftrightarrow} \frac{\omega}{s^2+\omega^2} \\
X(s) &= A\cos(\phi)\left(\frac{s}{s^2+\omega^2}\right) - A\sin(\phi)\left(\frac{\omega}{s^2+\omega^2}\right) \\
&= \frac{A\left[s\cos(\phi) - \omega\sin(\phi)\right]}{s^2+\omega^2} \\
Y(s) &= H(s)X(s) \\
Y(s) &= H(s)\frac{A\left[s\cos(\phi) - \omega\sin(\phi)\right]}{s^2+\omega^2}
\end{aligned}$$

Steady-State Sinusoidal Analysis Continued

$$
\begin{aligned}
Y(s) &= H(s)\frac{A\left[s\cos(\phi) - \omega\sin(\phi)\right]}{s^2+\omega^2} \\
&= \frac{k}{s-j\omega} + \frac{k^*}{s+j\omega} + \sum_{\ell=1}^{N}\frac{k_\ell}{s+p_\ell} \\
y(t) &= 2|k|\cos(\omega t + \angle k)u(t) + \sum_{\ell=1}^{N} k_\ell\,\mathrm{e}^{-pt}u(t) \\
&= y_{\mathrm{ss}}(t) + y_{\mathrm{tr}}(t) \\
y_{\mathrm{ss}}(t) &= \lim_{t\rightarrow\infty} y(t) \\
&= 2|k|\cos(\omega t + \angle k)
\end{aligned}
$$

Steady-State Sinusoidal Analysis Comments

If $x(t) = A\cos(\omega t + \phi)$,

$$y_{\text{ss}}(t) = \lim_{t\to\infty} y(t) = 2|k|\cos(\omega t + \angle k)$$

- If the input to an LTI system is sinusoidal,
  - The steady-state output is sinusoidal at the same frequency
  - The amplitude and phase of $y(t)$ differ from that of $x(t)$
- We applied this idea when we did phasor analysis
- But how is k related to $H(s)$, A, and ϕ?

Solving for the Complex Residue

$$\begin{aligned}
Y(s) &= H(s)\,\frac{A\,(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2} \\
&= \frac{k}{s-j\omega} + \frac{k^*}{s+j\omega} + \sum_{\ell=1}^{N}\frac{k_\ell}{s+p_\ell} \\
k &= H(s)\,\frac{A\,[s\cos(\phi) - \omega\sin(\phi)]}{s+j\omega}\bigg|_{s=+j\omega} \\
&= H(j\omega)\,\frac{A\,[j\omega\cos(\phi) - \omega\sin(\phi)]}{2j\omega} \\
&= H(j\omega)\,\frac{A\,[\cos(\phi) + j\sin(\phi)]}{2} \\
&= \tfrac{1}{2}H(j\omega)A\,\mathrm{e}^{j\phi}
\end{aligned}$$

Sinusoidal Steady-State Output

Since $H(j\omega)$ is complex, we can write it in polar form as

$$H(j\omega) = |H(j\omega)|\, \mathrm{e}^{j\angle H(j\omega)}$$

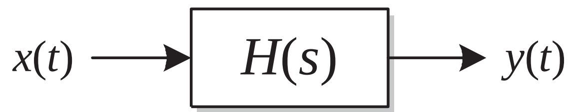
Then using the results of the previous slide, we have

$$k = \tfrac{1}{2} H(j\omega) A\, \mathrm{e}^{j\phi} \qquad = \tfrac{1}{2}|H(j\omega)| A\, \mathrm{e}^{j(\phi+\angle H(j\omega))}$$

$$|k| = \tfrac{1}{2}|H(j\omega)| A \qquad \angle k = \phi + \angle H(j\omega)$$

$$\begin{aligned} y_{\mathrm{ss}}(t) &= 2|k| \cos(\omega t + \angle k) \\ &= |H(j\omega)| A \cos\left(\omega t + \phi + \angle H(j\omega)\right) \end{aligned}$$

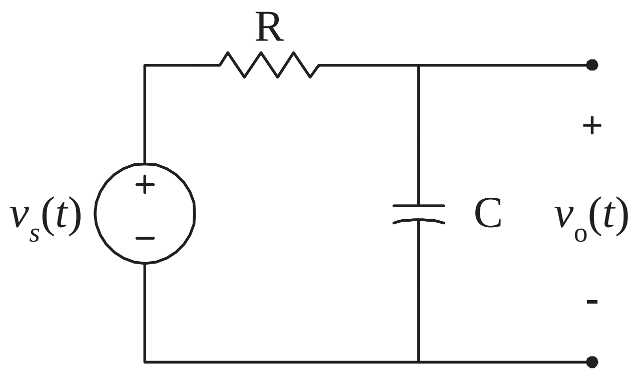
Sinusoidal Steady-State Output



$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ y_{\text{ss}}(t) &= |H(j\omega)| A \cos\left(\omega t + \phi + \angle H(j\omega)\right) \end{aligned}$$

- The input is sinusoidal
- The steady-state signal $y_{\text{ss}}(t)$ is also a sinusoid
  - Same frequency as $x(t)$: ω
  - Amplitude is scaled by $|H(j\omega)|$
  - The phase is shifted by $\angle H(j\omega)$
- If we know $H(s)$, we can easily find the steady-state solution for any sinusoidal input signal

Example 12: Steady-State Sinusoidal Analysis



Find the steady-state sinusoidal response to an input voltage of $v_s(t) = \cos(\omega t)$.

Example 12: Workspace

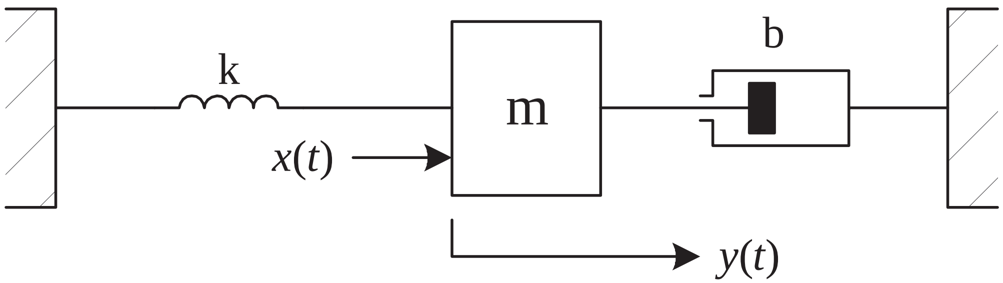
Steady-State Sinusoidal Analysis Comments

- We will study this in depth shortly
- There is analytical significance to how the magnitude and phase of $H(s)$ vary with $s = j\omega$

LTI Systems

- If we know the transfer function, we have sufficient information to calculate the output for any input
- This enables us to treat the circuit more abstractly as $H(s)$
- The transfer function may be for another type of system: mechanical, chemical, hydraulic, etc.
- Mathematically they are treated the same
- Field-specific analysis is used only to find $H(s)$

Example 13: Transfer Function Analysis

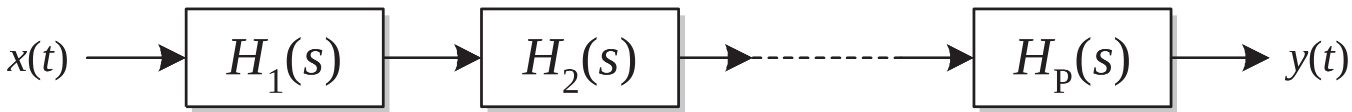


Find the transfer function for the linear system shown above. The external force $x(t)$ is the input to the system and the displacement $y(t)$ is the output. Find the transfer function.

Transfer Function Synthesis

- Thus far we have talked only about circuit *analysis*
- We now know several ways to solve for the output of a given system
- If there are zero initial conditions, then we can find the transfer function $H(s)$ of a given circuit
- Now we will discuss how to design a circuit that implements a given $H(s)$
- This is called **transfer function synthesis**
- There are many circuits that have the same transfer function

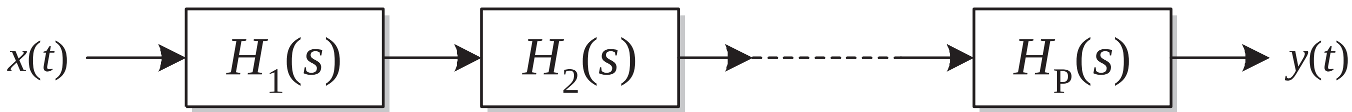
Cascade Transfer Function Synthesis



$$H(s) = \frac{N(s)}{D(s)} = H_1(s) \times H_2(s) \times \cdots \times H_P(s)$$

- There are many approaches to transfer function synthesis
- Will discuss how to specify $H(s)$ to meet the requirements for a given application later this term
- The most common (and perhaps easiest) approach to synthesis is to break $H(s)$ up into 1st (real poles) or 2nd (complex poles) order components
- Thus each component, $H_i(s)$ has either a 1st or 2nd order polynomial in the numerator and denominator

Cascade Transfer Function Synthesis Continued



- There are robust, standard circuits for implementing these low-order components
- The output of each transfer function is generated by an operational amplifier
- This is essential for the cascade synthesis to work (will explain later)
- Some of these 1st and 2nd order components are discussed in the text (Chapter 15)
- Others can be found in more advanced analog circuits texts
- You will use cascade synthesis in the first lab for ECE 203

Summary

- Circuits with a single input (independent source) and zero initial conditions can be represented generically by their transfer functions
- $H(s)$ is the Laplace transform of the system impulse response
- The output of the system is $y(t) = \mathcal{L}^{-1}\{H(s)X(s)\}$ for any causal input signal ($x(t) = 0$ for $t < 0$)
- For sinusoidal inputs, the output is also sinusoidal at the same frequency but amplified by $|H(j\omega)|$ and shifted in phase by $\angle H(j\omega)$
- Thus, transfer functions make sinusoidal steady-state analysis easy
- Generalization of phasors
- Transfer function analysis used for all types of LTI systems, not just circuits
- Can synthesize a transfer function using a cascade of 1st and 2nd order active circuits