Fundamentals of Signals Overview

- Definition
- Examples
- Energy and power
- Signal transformations
- Periodic signals
- Symmetry
- Exponential & sinusoidal signals
- Basis functions

Equation for a line

\[ x(t) = m(t - t_0) \]

- You will often need to quickly write an expression for a line given the slope and x-intercept
- Will use often when discussing convolution and Fourier transforms
- You should know how to apply this

Examples of Signals

Definition: an abstraction of any measurable quantity that is a function of one or more independent variables such as time or space.

Examples:
- A voltage in a circuit
- A current in a circuit
- The Dow Jones Industrial average
- Electrocardiograms
- \( A \sin(\omega t + \phi) \)
- Speech/music
- Force exerted on a shock absorber
- Concentration of Chlorine in a water supply
For most of this class we will use a broad definition of power and energy that applies to any signal $x(t)$ or $x[n]$.

**Instantaneous signal power**

\[ P(t) = |x(t)|^2 \quad P[n] = |x[n]|^2 \]

**Signal energy**

\[ E(t_0, t_1) = \int_{t_0}^{t_1} |x(t)|^2 \, dt \quad E(n_0, n_1) = \sum_{n=n_0}^{n_1} |x[n]|^2 \]

**Average signal power**

\[
\begin{align*}
P(t_0, t_1) &= \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |x(t)|^2 \, dt \\
P(n_0, n_1) &= \frac{1}{n_1 - n_0 + 1} \sum_{n=n_0}^{n_1} |x[n]|^2
\end{align*}
\]

We will encounter many types of signals.

- Some have infinite average power, energy, or both
- A signal is called an **energy signal** if $E_\infty < \infty$
- A signal is called a **power signal** if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither type
- A signal can not be both an energy signal and a power signal

**Discrete-time & Continuous-time**

- We will work with both types of signals
- Continuous-time signals
  - Will always be treated as a function of $t$
  - Parentheses will be used to denote continuous-time functions
  - Example: $x(t)$
  - $t$ is a continuous independent variable (real-valued)
- Discrete-time signals
  - Will always be treated as a function of $n$
  - Square brackets will be used to denote discrete-time functions
  - Example: $x[n]$
  - $n$ is an independent integer

**Signal Energy & Power Comments**

Usually, the limits are taken over an infinite time interval.

\[
\begin{align*}
E_\infty &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
E_\infty &= \sum_{n=-\infty}^{\infty} |x[n]|^2 \\
P_\infty &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \\
P_\infty &= \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x[n]|^2
\end{align*}
\]

We will encounter many types of signals.

- Some have infinite average power, energy, or both
- A signal is called an **energy signal** if $E_\infty < \infty$
- A signal is called a **power signal** if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither type
- A signal can not be both an energy signal and a power signal
Example 1: Energy & Power

Determine whether the energy and average power of each of the following signals is finite.

\[ x(t) = \begin{cases} 8 & \text{if } |t| < 5 \\ 0 & \text{otherwise} \end{cases} \]

\[ x[n] = j \]

\[ x[n] = A \cos(\omega n + \phi) \]

\[ x(t) = \begin{cases} e^{at} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ x[n] = e^{j\omega n} \]

Example 1: Workspace (1)

Signal Energy & Power Tips

- There are a few rules that can help you determine whether a signal has finite energy and average power:
  - Signals with finite energy have zero average power:
    \[ E_\infty < \infty \Rightarrow P_\infty = 0 \]
  - Signals of finite duration and amplitude have finite energy:
    \[ x(t) = 0 \text{ for } |t| > c \Rightarrow E_\infty < \infty \]
  - Signals with finite average power have infinite energy:
    \[ P_\infty > 0 \Rightarrow E_\infty = \infty \]
Signal Transformations

- Time shift: $x(t - t_0)$ and $x[n - n_0]$
  - If $t_0 > 0$ or $n_0 > 0$, signal is shifted to the right
  - If $t_0 < 0$ or $n_0 < 0$, signal is shifted to the left
- Time reversal: $x(-t)$ and $x[-n]$
- Time scaling: $x(\alpha t)$ and $x[\alpha n]$
  - If $\alpha > 1$, signal appears compressed
  - If $1 > \alpha > 0$, signal appears stretched

Use the signal shown above to draw the following: $x(-t)$, $x(t-1)$, $x(t)$, $x(t^2)$, $x(2t)$, $x(2-2t)$. 
Even & Odd Symmetry

\[ x_e(t) = \frac{1}{2} (x(t) + x(-t)) \]
\[ x_o(t) = \frac{1}{2} (x(t) - x(-t)) \]
\[ x(t) = x_e(t) + x_o(t) \]

- The symmetry of a signal under time reversal will be useful later when we discuss transforms
- A signal is even if and only if \( x(t) = x(-t) \)
- A signal is odd if and only if \( x(t) = -x(-t) \)
- \( \cos(k\omega_0 t) \) is an even signal
- \( \sin(k\omega_0 t) \) is an odd signal
- Any signal can be written as the sum of an odd signal and an even signal

Example 3: Even Symmetry

Draw the even component of the signal shown above.

Example 4: Odd Symmetry

Draw the odd component of the signal shown above.

Example 5: Even & Odd Symmetry

Show that the sum of the even and odd components of the signal is equal to the original signal graphically.
Periodic Signals

A signal is periodic if there is a positive value of $T$ or $N$ such that

$$x(t) = x(t + T) \quad x[n] = x[n + N]$$

- The fundamental period, $T_0$, for continuous-time signals is the smallest positive value of $T$ such that $x(t) = x(t + T)$
- The fundamental period, $N_0$, for discrete-time signals is the smallest positive integer of $N$ such that $x[n] = x[n + N]$.
- Signals that are not periodic are said to be aperiodic.

Exponential and Sinusoidal Signals

Exponential signals

$$x(t) = Ae^{at} \quad x[n] = Ae^{an}$$

where $A$ and $a$ are complex numbers.

- Exponential and sinusoidal signals arise naturally in the analysis of linear systems.
- Example: simple harmonic motion that you learned in physics.
- There are several distinct types of exponential signals:
  - $A$ and $a$ real
  - $A$ and $a$ imaginary
  - $A$ and $a$ complex (most general case)
Example 7: MATLAB Code

```matlab
fs = 500; % Sample rate (Hz)
t = -10:1/fs:30; % Time index (s)
a = j; A = 1;
y = A*exp(a*t);
h = plot3(t,imag(y),real(y),'b');hold on;
h = plot3(t,ones(size(t)),real(y),'r');
h = plot3(t,imag(y),-ones(size(t)),'g');hold off;
grid on;
xlabel('Time (s)');ylabel('Imaginary Part');zlabel('Real Part');title('Complex:Blue Real:Red Imaginary:Green');view(27.5,22);
```

Sinusoidal Exponential Signal Comments

\[ x(t) = Ae^{at} = A(e^{a})^t = A\alpha^t \quad x[n] = Ae^{an} = A(e^{a})^n = A\alpha^n \]

When \( a \) is imaginary, then Euler's equation applies:

\[ e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \]
\[ e^{j\omega n} = \cos(\omega n) + j\sin(\omega n) \]

- Since \( |e^{j\omega t}| = 1 \), this looks like a coil in a plot of the complex plane versus time
- \( e^{j\omega t} \) is Periodic with fundamental period \( T = \frac{2\pi}{\omega} \)
- Real part is sinusoidal: \( \text{Re}\{Ae^{j\omega t}\} = A\cos(\omega t) \)
- Imaginary part is sinusoidal: \( \text{Im}\{Ae^{j\omega t}\} = A\sin(\omega t) \)
- These signals have infinite energy, but finite (constant) average power, \( P_\infty \)

Example 7: \( Ae^{at} \), \( A = 1 \) and \( a = j \)

Sinusoidal Exponential Harmonics

- In order for \( e^{j\omega t} \) to be periodic with period \( T \), we require that
  \[ e^{j\omega t} = e^{j\omega(t+T)} = e^{j\omega t}e^{j\omega T} \text{ for all } t \]
- This implies \( e^{j\omega T} = 1 \) and therefore
  \[ \omega T = 2\pi k \text{ where } k = 0, \pm 1, \pm 2, \ldots \]
- There is more than one frequency \( \omega \) that satisfies the constraint \( x(t) = x(t+T) \) where \( T = \frac{2\pi k}{\omega} \)
- The fundamental frequency is given by \( k = 1 \):
  \[ \omega_0 = \frac{2\pi}{T_0} \]
- The other frequencies that satisfy this constraint are then integer multiples of \( \omega_0 \)
Sinusoidal Exponential Harmonics Continued

A harmonically related set of complex exponentials is a set of exponentials with fundamental frequencies that are all multiples of a single positive frequency $\omega_0$

$$\phi_k(t) = e^{j k \omega_0 t} \text{ where } k = 0, \pm 1, \pm 2, \ldots$$

- For $k = 0$, $\phi_k(t)$ is a constant
- For all other values $\phi_k(t)$ is periodic with fundamental frequency $|k| \omega_0$
- This is consistent with how the term harmonic is used in music
- Sinusoidal harmonics will play a very important role when we discuss Fourier series and periodic signals

Example 9: MATLAB Code

```matlab
n = -10:40; % Time index
omega = 2*pi/20; % Frequency (radians/sample)
N = 4;
for cnt = 0:N,
    subplot(N+1,1,cnt+1); phi = exp(j*cnt*omega*n);
    h = stem(n,real(phi));
    set([h(1)],'Marker','.'); ylim([-1.1 1.1]); ylabel(sprintf('\phi_%d',cnt));
    set(gca,'YGrid','On')
    if cnt~=N,
        set(gca,'XTickLabel',[]);
    end;
end;
xlabel('Time (n)');
figure;
t = -10:0.01:40; % Time index
omega = 0.05*2*pi; % Frequency (radians/sec) - 0.05 Hz
N = 4;
for cnt = 0:N,
    subplot(N+1,1,cnt+1); phi = exp(j*cnt*omega*t);
    h = plot(t,real(phi));
    ylim([-1.1 1.1]);
    ylabel(sprintf('\phi_%d',cnt)); grid on;
    if cnt~=N,
        set(gca,'XTickLabel',[]);
    end;
end;
xlabel('Time (s)');
```

Example 8: Continuous-Time Harmonics
Damped Complex Sinusoidal Exponentials

\[ x(t) = Ae^{at} \quad x[n] = Ae^{an} \]

- When \( a \) is complex, these become damped sinusoidal exponentials
- Let \( a = \alpha + j\omega \). Then
  \[ x(t) = Ae^{at} = (Ae^{\alpha t}) \times e^{j\omega t} \quad x[n] = Ae^{an} = (Ae^{\alpha n}) \times e^{j\omega n} \]
- Thus, these are equivalent to multiplying an complex sinusoid by a real exponential

Example 10: MATLAB Code

\begin{verbatim}
  n = -10:40; \% Time index
  C = 1; subplot(2,1,1);
  a = 0.1 + j*0.5;
  y = real(C*exp(a*n)); \% Growing exponential
  t = min(n):0.1:max(n);
  h = plot(t, real(C*exp(real(a)*t)),t,-real(C*exp(real(a)*t)));
  set(h,'Color',[0.5 0.5 0.5]);
  hold on;
  h = stem(n,y);
  set(h(1),'Marker','.');
  hold off;
  box off;
  grid on;
  ylim([-50 50]);
  ylabel('Real Part');

  subplot(2,1,2);
  a = -0.1 + j*0.5;
  y = real(C*exp(a*n)); \% Growing exponential
  t = min(n):0.1:max(n);
  h = plot(t, real(C*exp(real(a)*t)),t,-real(C*exp(real(a)*t)));
  set(h,'Color',[0.5 0.5 0.5]);
  hold on;
  h = stem(n,y);
  set(h(1),'Marker','.');
  hold off;
  box off;
  grid on;
  ylim([-2 2]);
  xlabel('Time (n)');
  ylabel('Real Part');
\end{verbatim}
The discrete-time unit step is defined as

\[ u[n] = \begin{cases} 
  0, & n < 0 \\
  1, & n \geq 0 
\end{cases} \]

The discrete-time unit impulse is defined as

\[ \delta[n] = \begin{cases} 
  0, & n \neq 0 \\
  1, & n = 0 
\end{cases} \]

- Sometimes called the unit sample
- Also called the Kroneker delta
- Note that \( \delta[n] \) has even symmetry so \( \delta[n] = \delta[-n] \)

The discrete-time basis functions relate \( \delta[n] \) and \( u[n] \) as follows:

\[ \delta[n] = u[n] - u[n-1] \]

\[ u[n] = \sum_{k=-\infty}^{n} \delta[k] \]

\[ u[n] = \sum_{k=0}^{n} \delta[n-k] \]

The unit impulse can be used to sample a discrete-time signal \( x[n] \):

\[ x[0] = \sum_{k=-\infty}^{\infty} x[k] \delta[k] \]

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \]

This ability to use the unit impulse to extract a single value of \( x[n] \) through multiplication will play an important role later in the term.
**Continuous-Time Unit Impulse**

\[ u(t) \triangleq \begin{cases} 
0 & t < 0 \\
1 & t > 0 
\end{cases} \]

- Sometimes known as the **Heaviside function**
- Discontinuous at \( t = 0 \)
- \( u(0) \) is not defined
- Not of consequence because it is undefined for an infinitesimal period of time

\[ \delta(t) \triangleq \lim_{\epsilon \to 0} \delta_{\epsilon}(t) \]

- \( \delta_e(t) \triangleq \frac{\text{d}u_e(t)}{\text{d}t} \)
- As \( \epsilon \to 0 \),
  - \( u_e(t) \to u(t) \)
  - \( \delta_e(t) \) for \( t = 0 \) becomes very large
  - \( \delta_e(t) \) for \( t \neq 0 \) becomes zero
- \( \delta(t) \triangleq \lim_{\epsilon \to 0} \delta_{\epsilon}(t) \)

**Continuous-Time Unit Step**

\[ u(t) \]

- Sometimes known as the **Heaviside function**
- Discontinuous at \( t = 0 \)
- \( u(0) \) is not defined
- Not of consequence because it is undefined for an infinitesimal period of time

\[ \int_{-\epsilon}^{\epsilon} \delta(t) \, dt = 1 \text{ for any } \epsilon > 0 \]

- Also known as the **Dirac delta** function
- Is zero everywhere except zero
- The impulse integral serves as a measure of the impulse amplitude
- Drawn as an arrow with unit height
- \( 5\delta(t) \) would be drawn as an arrow with height of 5

**Unit Step for Switches**

- \( u(t) \) useful for representing the opening or closing of switches
- We will often solve for or be given initial conditions at \( t = 0 \)
- We can then represent independent sources as though they were immediately applied at \( t = 0 \). More later.
Continuous-Time Unit Impulse Comments

\[ \delta(t) = \begin{cases} 
0 & t \neq 0 \\
\infty & t = 0 
\end{cases} \]

- The impulse should be viewed as an idealization
- Real systems with finite inertia do not respond instantaneously
- The most important property of an impulse is its area
- Most systems will respond nearly the same to sharp pulses regardless of their shape - if
  - They have the same amplitude (area)
  - Their duration is much briefer than the system’s response
- The idealized unit impulse is short enough for any system

Unit Impulse Properties

\[ \delta(t)x(t) = \delta(t)x(0) \]
\[ \delta(at) = \frac{1}{|a|}\delta(t) \]
\[ \delta(-t) = \delta(t) \]
\[ \delta(t) = \frac{du(t)}{dt} \]
\[ u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \]

Unit Impulse Sampling Property

\[ \int_{-\infty}^{+\infty} x(t)\delta(t) \, dt = \int_{-\infty}^{+\infty} x(0)\delta(t) \, dt \]
\[ = x(0) \int_{-\infty}^{+\infty} \delta(t) \, dt \]
\[ = x(0) \]

Similarly,

\[ \int_{-\infty}^{+\infty} x(t)\delta(t - t_0) \, dt = \int_{-\infty}^{+\infty} x(t_0)\delta(t - t_0) \, dt \]
\[ = x(t_0) \int_{-\infty}^{+\infty} \delta(t - t_0) \, dt \]
\[ = x(t_0) \]

Unit Impulse Sampling Property

\[ x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau) \, d\tau \]

- This integral does not appear to be useful
- It will turn out to be very useful
- It states that \( x(t) \) can be written as a linear combination of scaled and shifted unit impulses
- This will be a key concept when we discuss convolution next week
Basis Function Relationships

\[
\begin{align*}
  u(t) &= \int_{-\infty}^{t} \delta(\tau) \, d\tau \\
  r(t) &= \int_{-\infty}^{t} u(\tau) \, d\tau \\
  \frac{du(t)}{dt} &= \delta(t) \\
  \frac{dr(t)}{dt} &= u(t) \\
  \int_{-\infty}^{t} u(\tau) \, d\tau &= r(t) \\
  \int_{-\infty}^{t} r(\tau) \, d\tau &= \frac{1}{2} r(t)^2
\end{align*}
\]

- If we can write a signal \( x(t) \) in terms of \( u(t) \) and \( r(t) \), it is easy to find the derivative.
- Similarly, it is easy to integrate.

Example 12: Continuous-Time Unit-Ramp

\[
r(t) = \begin{cases} 
  0 & t \leq 0 \\
  t & t \geq 0
\end{cases}
\]

What is the first derivative?

Example 13: Continuous-Time Unit-Ramp Integral

What is the integral of the unit ramp?

Basis Functions Translated

- Can write simple expressions for the functions translated in time.
- Can scale the amplitude.
- Any piecewise linear signal can be written in terms of basis functions.
- This makes it easy to calculate derivatives and integrals.
- Will not discuss how this term.
- Sufficient to know it can be done.