Laplace Transforms

- Definition
- Region of convergence
- Useful properties
- Inverse & partial fraction expansion
- Distinct, complex, & repeated poles
- Applied to linear constant-coefficient ODE’s
Laplace Transform Motivation

- In ECE 221, you learned
  - DC circuit analysis
  - Transient response (limited to simple RL & RC circuits)
  - Sinusoidal steady-state response (Phasors)
- We did not learn how to find the total response (transient and steady-state) to an arbitrary waveform
- The Laplace transform enables us to do this
- Circuit elements limited to resistors, capacitors, inductors, transformers, op amps, and ideal sources until ECE 321
Why are we studying the Laplace transform?

- Makes analysis of circuits
  - Easier than working with multiple differential equations
  - More general than the types of analysis we discussed in ECE 221
- Used extensively in
  - Controls (ECE 311)
  - Communications
  - Signal Processing
  - Analog circuits (ECE 32X sequence)
- Expected to know for interviews
- Gives you *insight in circuit analysis and design*
Given \( v_o(0) = 0 \), solve for \( v_o(t) \) for \( t \geq 0 \).

\[
\begin{align*}
\quad v_o(t) & = \frac{1}{2} e^{-t/0.001} + \frac{1}{\sqrt{2}} \sin(1000t - 45^\circ) \\
& = v_{tr}(t) + v_{ss}(t) \\
v_{tr}(t) & = \frac{1}{2} e^{-t/0.001} \\
v_{ss}(t) & = \frac{1}{\sqrt{2}} \sin(1000t - 45^\circ)
\end{align*}
\]
\[ v_o(t) = \frac{1}{2} e^{-t/0.001} + \frac{1}{\sqrt{2}} \sin(1000t - 45^\circ) \]
\[ = v_{tr}(t) + v_{ss}(t) \]
Laplace Transform for ODE’s

- Relationship of a voltage (or current) in a linear circuit to any other voltage (or current) is defined by a linear, time-invariant constant-coefficient ordinary differential equation (ODE).
- Describes the behavior of many types of systems: Electrical, Mechanical, Chemical, Biological, etc.
- Laplace transform is an easier approach than applying standard techniques of differential equations or convolution.
Approach

• We will begin with a thorough discussion of the Laplace transform
• The elegance and simplicity of using this approach for circuit analysis will not become apparent for several lectures
• We will spend a lot of time on this topic
• Bear with me
**Laplace Transform Definition**

\[ \mathcal{L} \{ x(t) \} = X(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st} \, dt \]

- Transform will be written with an upper-case letter
- Defined from \( 0^- \) to include impulses at \( t = 0 \)
- \( s = \sigma + j\omega \) is a complex variable
- \( s \) has units of inverse seconds (\( s^{-1} \))
- Known as the **one-sided** (unilateral) Laplace transform
- There is also a **two-sided** (bilateral) version:
  \[ X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} \, dt \]
- We will only work with the one-sided version
  + Easier to obtain the transient response
  + Consistent with common practice
  - Ignores \( x(t) \) for \( t < 0 \)
Laplace Transform Convergence

- The Laplace transform does not converge to a finite value for all signals and all values of $s$
- Does converge for all signals we will be interested in
  - Sinusoids
  - $e^{-at}u(t)$ for any real $|a| < \infty$
  - $\delta(t)$
- The values of $s$ for which the Laplace transform converges is called the region of convergence (ROC)
- Will not discuss in detail this term, but may see this in other classes on linear systems
- See *Signals and Systems* chapter for more information
Example 1: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = u(t)$. What is the region of convergence? What is the transform of $x(t) = 1$?
Example 2: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = e^{-at}u(t)$. What is the region of convergence? What is the transform of $x(t) = e^{-at}$?
Example 3: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = \delta(t)$. What is the region of convergence?
Example 4: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = \cos(\omega t)u(t)$. What is the region of convergence? What is the Laplace transform of $\cos(\omega t)$?
Example 4: Workspace
Laplace Transform Properties

• The Laplace transform has many important properties
• We need to know these for at least three reasons
  – Improves our understanding of the transform
  – Enables us to find the transform more easily
  – Enables us to find the inverse transform more easily
• Will use the following notation for Laplace transform pairs

\[ x(t) u(t) \Leftrightarrow X(s) \]

\[ X(s) = \mathcal{L}\{x(t)\} \]

\[ x(t) u(t) = \mathcal{L}^{-1}\{X(s)\} \]
Linearity

\[ X_1(s) = \mathcal{L}\{x_1(t)\} \]
\[ X_2(s) = \mathcal{L}\{x_2(t)\} \]

then you should be able to show that

\[ [a_1 x_1(t) + a_2 x_2(t)] u(t) \overset{\mathcal{L}}{\longleftrightarrow} a_1 X_1(s) + a_2 X_2(s) \]

Example: Find the Laplace transform of \( x(t) = 5\delta(t) - 2\cos 5t \).

\[ \mathcal{L}\{\delta(t)\} = 1 \]
\[ \mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2} \]
\[ \mathcal{L}\{5\delta(t) - 2\cos 5t\} = 5(1) - 2 \left( \frac{s}{s^2 + 5^2} \right) \]
\[ = 5 - \frac{2s}{s^2 + 25} \]
Scaling

Given $X(s) = \mathcal{L}\{x(t)\}$, what is $\mathcal{L}\{x(at)\}$ for $a > 0$?

$$\mathcal{L}\{x(at)\} = \int_{0^-}^{\infty} x(at)e^{-st} \, dt$$

$$\tau = at \quad \quad \quad \quad \quad t = \frac{\tau}{a}$$

$$d\tau = a \, dt \quad \quad \quad \quad \quad dt = \frac{1}{a} \, d\tau$$

$$\mathcal{L}\{x(at)\} = \frac{1}{a} \int_{0^-}^{\infty} x(\tau)e^{-s\frac{\tau}{a}} \, d\tau$$

$$= \frac{1}{a} \int_{0^-}^{\infty} x(\tau)e^{-\left(\frac{s}{a}\right)\tau} \, d\tau$$

$$x(at)u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right)$$
Translation in Time

Given \( X(s) = \mathcal{L}\{x(t)\} \), what is \( \mathcal{L}\{x(t - t_0)u(t - t_0)\} \) for \( t_0 > 0 \)?

\[
\mathcal{L}\{x(t - t_0)u(t - t_0)\} = \int_{0^-}^\infty x(t - t_0)u(t - t_0)e^{-st}\,dt
\]

\[
\tau = t - t_0
\]

\[
d\tau = dt
\]

\[
t = \tau + t_0
\]

\[
\mathcal{L}\{x(t - t_0)u(t - t_0)\} = \int_{-t_0}^\infty x(\tau)u(\tau)e^{-s(\tau + t_0)}\,d\tau
\]

\[
= \int_{-t_0}^\infty x(\tau)u(\tau)e^{-s(\tau + t_0)}\,d\tau
\]

\[
= e^{-st_0} \int_{0^-}^\infty x(\tau)e^{-s\tau}\,d\tau
\]

\[
x(t - t_0)u(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s)
\]
Translation in Frequency

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the inverse Laplace transform of $X(s + s_0)$?

$$X(s + s_0) = \int_{0^-}^{\infty} x(t) e^{-(s+s_0)t} \, dt$$

$$= \int_{0^-}^{\infty} (x(t) e^{-s_0 t}) e^{-st} \, dt$$

$$= \mathcal{L}\{ e^{-s_0 t} x(t) \}$$

$$\mathcal{L}^{-1}\{X(s + s_0)\} = e^{-s_0 t} x(t) u(t)$$

$$e^{-s_0 t} x(t) u(t) \Longleftrightarrow X(s + s_0)$$
**Time Differentiation**

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the Laplace transform of $\dot{x}(t)$?

\[
\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^{-}}^{\infty} \frac{dx(t)}{dt} e^{-st} \, dt
\]

\[
u = e^{-st} \quad \text{du} = -se^{-st} \, dt
\]

\[
v = x(t)
\]

\[
\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^{-}}^{\infty} u \, dv = uv|_{0^{-}}^{\infty} - \int_{0^{-}}^{\infty} v \, du
\]

\[
= e^{-st} x(t)|_{0^{-}}^{\infty} - \int_{0^{-}}^{\infty} x(t) (-se^{-st}) \, dt
\]

\[
= (0 - x(0^{-})) + s \int_{0^{-}}^{\infty} x(t) e^{-st} \, dt
\]

\[
\frac{dx(t)}{dt} \quad u(t) \iff \mathcal{L} \quad sX(s) - x(0^{-})
\]
Time Differentiation Continued

In general

\[ \frac{d^n x(t)}{dt^n} u(t) \Leftrightarrow s^n X(s) - s^{n-1} x(0^-) - \cdots - s^0 \left. \frac{d^{n-1} x(t)}{dt^{n-1}} \right|_{t=0^-} \]

If all of the initial conditions are zero,

\[ \frac{d^n x(t)}{dt} u(t) \Leftrightarrow s^n X(s) \]
Time Integration

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the Laplace transform of $\int_{0-}^{t} x(\tau) \, d\tau$?

\[
\mathcal{L}\left\{\int_{0}^{t} x(\tau) \, d\tau\right\} = \int_{0-}^{\infty} \left(\int_{0}^{t} x(\tau) \, d\tau\right) e^{-st} \, dt
\]

\[
u = \int_{0}^{t} x(\tau) \, d\tau \quad \text{du} = x(t) \, dt
\]

\[
v = \frac{-1}{s} e^{-st}
\]

\[
\mathcal{L}\left\{\int_{0}^{t} x(\tau) \, d\tau\right\} = \left(\int_{0}^{t} x(\tau) \, d\tau\right) \frac{-1}{s} e^{-st} \bigg|_{0}^{\infty} - \int_{0-}^{\infty} \frac{-1}{s} e^{-st} x(t) \, dt
\]

\[
= (0 - 0) + \frac{1}{s} \int_{0-}^{\infty} x(t) e^{-st} \, dt
\]

\[
\int_{0}^{t} x(\tau) \, d\tau \iff \frac{1}{s} X(s)
\]
Other Properties

- Other properties of the Laplace transform are derived in the text
- See Table 15.1 (page 687) of the electric circuits text
- Common Laplace transform pairs are listed in Table 15.2 (Page 687)
- You should put copies of these tables on your notes that you bring to the exams
Example 5: Laplace Transform Properties

Find the Laplace transform of \( t u(t) \).
Example 6: Laplace Transform Properties

Find the Laplace transform of

\[ \dot{r}(t) \triangleq \frac{dr(t)}{dt} \]
Example 7: Laplace Transform Properties

Find the Laplace transform of $e^{-at} \cos(\omega t) u(t)$. Hint:

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$
Inverse Laplace Transform Overview

\[ \mathcal{L}^{-1} \{X(s)\} = \frac{1}{j2\pi} \int_{\sigma_1-j\infty}^{\sigma_1+j\infty} X(s) e^{st} \, ds \]

- The inverse Laplace transform is given above
- \( \sigma_1 \) is such that the integral is taken over a line in the region of convergence
- Very difficult to apply directly
- We will use a different approach
- Convert \( X(s) \) to a form such that we can easily find the inverse
Inverse Laplace Transform Example

\[ X(s) = \frac{s + 8}{s(s + 2)} = \frac{4}{s} - \frac{3}{s + 2} \]

Since we know

\[ u(t) \quad \mathcal{L} \quad \frac{1}{s} \]

\[ e^{-at}u(t) \quad \mathcal{L} \quad \frac{1}{s + a} \]

\[ a_1x_1(t)u(t) + a_2x_2(t)u(t) \quad \mathcal{L} \quad a_1X_1(s) + a_2X_2(s) \]

we know that the inverse Laplace transform of \( X(s) \) is

\[ \mathcal{L}^{-1} \{ X(s) \} = 4u(t) - 3e^{-2t}u(t) \]
Partial Fraction Expansion

A critical step in the previous example was finding following equation:

\[ X(s) = \frac{s + 8}{s(s + 2)} = \frac{4}{s} - \frac{3}{s + 2} \]

- In general, this can be done by partial fraction expansion
- In practice, we can do this using your calculators, MATLAB, or a similar tool
In general, most functions will have a general form

\[ X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{N} a_k s^k}{\sum_{k=0}^{M} b_k s^k} \]

- \( N(s) \) and \( D(s) \) are polynomials in \( s \).
- The roots of \( N(s) = 0 \) are called zeros of \( X(s) \).
- The roots of \( D(s) = 0 \) are called poles of \( X(s) \).
- To find \( x(t) = \mathcal{L}^{-1} \{ X(s) \} \), we need to
  1. Find the poles of \( X(s) \)
  2. Apply partial fraction expansion (via MATLAB)
  3. Find the inverse of each term by table lookup
PFE: Distinct Poles

\[ X(s) = \frac{N(s)}{D(s)} \]

\[ = \frac{N(s)}{(s + p_1)(s + p_2)\ldots(s + p_n)} \]

\[ = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \ldots + \frac{k_n}{s + p_n} \]

\[ k_i = (s + p_i)X(s)|_{s=-p_i} \]

- A distinct pole is a unique root of \( D(s) = 0 \)
- The coefficients \( k_i \) are called the \textbf{residues} of \( X(s) \)
- To find \( k_i \) multiply both sides by \( (s + p_i) \) and evaluate at \( s = -p_i \)
Example 8: Partial Fraction Expansion

Given \( X(s) = \mathcal{L}\{x(t)\} \), find \( x(t) \).

\[
X(s) = \frac{5s + 29}{s^3 + 8s^2 + 19s + 12}
\]

\[\Rightarrow [r,p,k] = \text{residue([5 29],[1 8 19 12])}\]

\( r = 3.0000, -7.0000, 4.0000 \)

\( p = -4.0000, -3.0000, -1.0000 \)

\( k = [] \)
Example 8: Workspace
PFE: Distinct Complex Poles Method 1

\[ X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + \alpha - j\beta)(s + \alpha + j\beta)} = \frac{k_1}{s + \alpha - j\beta} + \frac{k_1^*}{s + \alpha + j\beta} \]

\[ k_1 = \left. (s + \alpha - j\beta)X(s) \right|_{s=-\alpha+j\beta} \]

- There are two methods for handling complex poles
- Often the residues of \( X(s) \) will be complex
- In this case, the complex roots of \( X(s) \) will be in complex conjugate pairs
- The residues will also be complex conjugate pairs
\[ X(s) = \frac{N(s)}{D(s)} \]
\[ = \frac{k_1 s + k_2}{s^2 + as + b} + R(s) \]
\[ = \frac{k_1 s + k_1 \alpha + k_2 - k_1 \alpha}{(s + \alpha)^2 + \beta^2} + R(s) \]
\[ = \frac{k_1 (s + \alpha) + k_2 - k_1 \alpha}{(s + \alpha)^2 + \beta^2} + R(s) \]
\[ = \frac{c_1 (s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{c_2 \beta}{(s + \alpha)^2 + \beta^2} + R(s) \]
\[ c_1 = k_1 \]
\[ c_2 = \frac{k_2 - k_1 \alpha}{\beta} \]
\[ x(t) = c_1 e^{-\alpha t} \cos(\beta t) u(t) + c_2 e^{-\alpha t} \sin(\beta t) u(t) + \mathcal{L}^{-1} \{ R(s) \} \]
Example 9: Distinct Complex Poles

Given \( X(s) = \mathcal{L}\{x(t)\} \), find \( x(t) \) using both methods of handling complex poles.

\[
X(s) = \frac{2}{s^3 + 2s^2 + 2s}
\]

\[ r = -0.5000 + 0.5000i, -0.5000 - 0.5000i, 1.0000 \]
\[ p = -1.0000 + 1.0000i, -1.0000 - 1.0000i, 0 \]
\[ k = [] \]
PFE: Useful Transforms

\[
\frac{k}{s + a} \iff ke^{-at}u(t)
\]

\[
\frac{k}{(s + a)^2} \iff kte^{-at}u(t)
\]

\[
\frac{k}{s + \alpha - j\beta} + \frac{k^*}{s + \alpha + j\beta} \iff 2|k|e^{-\alpha t}\cos(\beta t + \theta_k)u(t)
\]

\[
\frac{k}{(s + \alpha - j\beta)^2} + \frac{k^*}{(s + \alpha + j\beta)^2} \iff 2t|k|e^{-\alpha t}\cos(\beta t + \theta_k)u(t)
\]

where \( k = |k|\angle \theta_k \)

- You solve for \( k \) just as for a single pole (residue)
- Can also complete the square, as described in the textbooks
PFE: Repeated Poles

\[ X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p)^n} \]
\[ = \frac{k_1}{s + p} + \frac{k_2}{(s + p)^2} + \cdots + \frac{k_n}{(s + p)^n} \]
\[ k_{n-m} = \frac{1}{m!} \frac{d^m}{ds^m} (s + p)^n X(s) \bigg|_{s=-p} \]
\[ \mathcal{L}^{-1} \left\{ \frac{1}{(s + p)^n} \right\} = \frac{1}{(n-1)!} t^{n-1} e^{-pt} u(t) \]

- You can apply the equations above to handle repeated poles
- The \textbf{algebraic method} is usually easier
Example 10: Repeated Poles

Given $X(s) = \mathcal{L}\{x(t)\}$, find $x(t)$.

$$X(s) = \frac{s - 2}{s(s + 1)^3}$$

$\Rightarrow [r,p,k] = \text{residue}([1\ -2],\text{poly}([0\ -1\ -1\ -1]))$

$r = 2.0000, 2.0000, 3.0000, -2.0000$

$p = -1.0000, -1.0000, -1.0000, 0$

$k = []$
Example 10: Workspace
PFE: Improper Rational Functions

\[ X(s) = \frac{N(s)}{D(s)} = A(s) + \frac{B(s)}{C(s)} \]

- If \( X(s) \) is an improper rational function, you must convert it to an expression that contains a proper rational function
- Will not explain conversion
- Key point: if the order of \( N(s) \) is greater than or equal to the order of \( D(s) \), you cannot apply PFE directly
- MATLAB’s residue will find the coefficients of \( A(s) \) as part of the partial fraction expansion
Solving Ordinary Differential Equations

\[
\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t)
\]

\[
\sum_{k=0}^{N} a_k s^k Y(s) - \sum_{k=0}^{N} \sum_{\ell=1}^{k} s^{k-\ell} y^{(\ell-1)}(0^-) = X(s)
\]

\[
Y(s) = \frac{X(s) + \sum_{k=0}^{N} \sum_{\ell=1}^{k} s^{k-\ell} y^{(\ell-1)}(0^-)}{\sum_{k=0}^{N} a_k s^k}
\]

If \( \mathcal{L}\{x(t)\} \) is a rational function of \( s \), then the linear ordinary differential equation shown above can be solved (more easily) using the Laplace Transform.
Example 11: Solving ODE’s

Solve the following ODE for $y(t)$ given that $y(0^-) = \dot{y}(0^-) = 0$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 20 \cos(2t)u(t) + \frac{1}{2} \sin(2t)u(t)$$

Hint:

$$\gg [r,p,k] = \text{residue}([20 1],[\text{conv}([1 3 2],[1 0 4])])$$

$r = 4.8750, -0.5375 - 1.4875i, -0.5375 + 1.4875i, -3.8000$

$p = -2.0000, -0.0000 + 2.0000i, -0.0000 - 2.0000i, -1.0000$

$k = []$

$$\gg [\text{abs}(r) \text{ angle}(r)*180/\pi]$$

$ans = 4.8750 0, 1.5816 -109.8670, 1.5816 109.8670, 3.8000 180.0000,$
Example 11: Workspace 2
Summary

- The Laplace transform can be used to solve ordinary differential equations
- This includes circuits and many other linear time-invariant systems
- We use the one-sided Laplace transform
- The inverse of this transform is always 0 for $t < 0$
- We usually solve for the inverse by using known transform pairs and the properties of the transform
- This topic is covered in both books