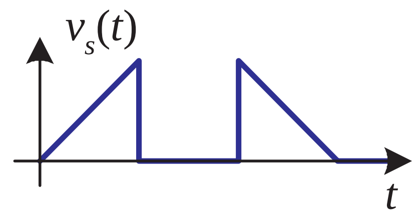
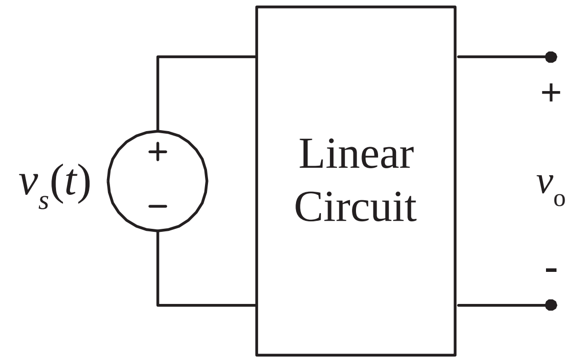
Laplace Transforms

- Definition
- Region of convergence
- Useful properties
- Inverse & partial fraction expansion
- Distinct, complex, & repeated poles
- Applied to linear constant-coefficient ODE's

Laplace Transform Motivation





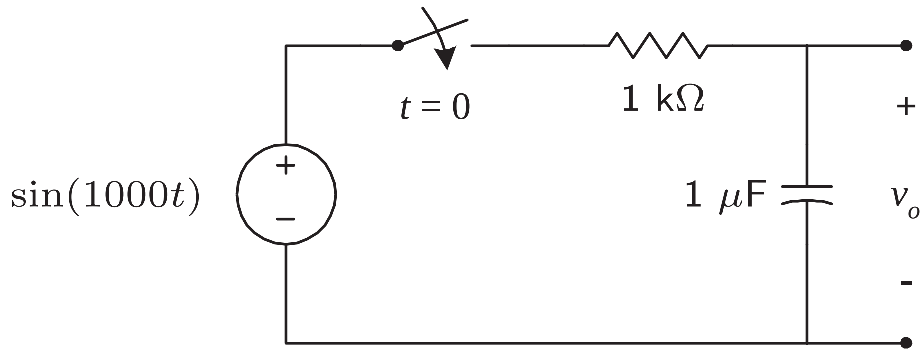
- In ECE 221, you learned
  - DC circuit analysis
  - Transient response (limited to simple RL & RC circuits)
  - Sinusoidal steady-state response (Phasors)
- We did not learn how to find the total response (transient and steady-state) to an arbitrary waveform
- The Laplace transform enables us to do this
- Circuit elements limited to resistors, capacitors, inductors, transformers, op amps, and ideal sources until ECE 321

Laplace Transform Motivation Continued

Why are we studying the Laplace transform?

- Makes analysis of circuits
  - Easier than working with multiple differential equations
  - More general than the types of analysis we discussed in ECE 221
- Used extensively in
  - Controls (ECE 311)
  - Communications
  - Signal Processing
  - Analog circuits (ECE 32X sequence)
- Expected to know for interviews
- Gives you *insight in circuit analysis and design*

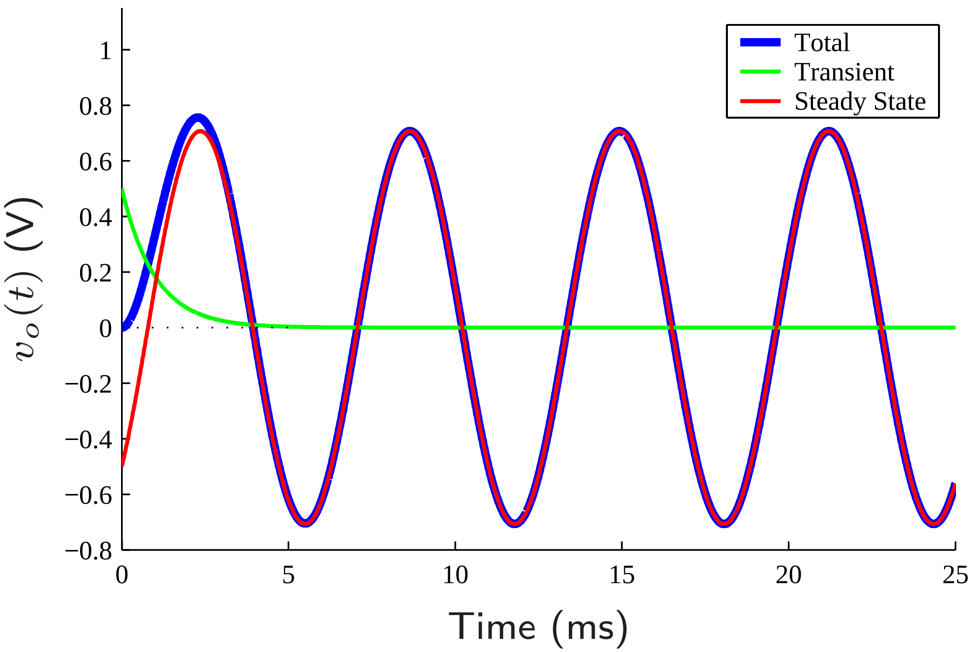
Laplace Transform Analysis Illustration



Given $v_o(0) = 0$, solve for $v_o(t)$ for $t \geq 0$.

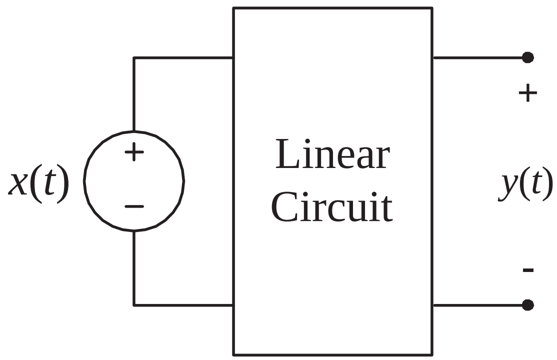
$$\begin{aligned} v_o(t) &= \frac{1}{2}\mathrm{e}^{-t/0.001} + \frac{1}{\sqrt{2}}\sin(1000t - 45^\circ) \\ &= v_{\mathsf{tr}}(t) + v_{\mathsf{ss}}(t) \\ v_{\mathsf{tr}}(t) &= \frac{1}{2}\mathrm{e}^{-t/0.001} \\ v_{\mathsf{ss}}(t) &= \frac{1}{\sqrt{2}}\sin(1000t - 45^\circ) \end{aligned}$$

Laplace Transform Analysis Illustration Continued



$$
\begin{aligned}
v_o(t) &= \tfrac{1}{2}\mathrm{e}^{-t/0.001} + \tfrac{1}{\sqrt{2}}\sin(1000t - 45°) \\
&= v_{\mathsf{tr}}(t) + v_{\mathsf{ss}}(t)
\end{aligned}
$$

Laplace Transform for ODE's



$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

- Relationship of a voltage (or current) in a linear circuit to any other voltage (or current) is defined by a linear, time-invariant constant-coefficient ordinary differential equation (ODE)
- Describes the behavior of many types of systems: Electrical, Mechanical, Chemical, Biological, etc.
- Laplace transform is an easier approach than applying standard techniques of differential equations or convolution

Approach

- We will begin with a thorough discussion of the Laplace transform
- The elegance and simplicity of using this approach for circuit analysis will not become apparent for several lectures
- We will spend a lot of time on this topic
- Bear with me

Laplace Transform Definition

$$\mathcal{L}\{x(t)\} = X(s) \triangleq \int_{0^-}^{\infty} x(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

- Transform will be written with an upper-case letter
- Defined from 0^- to include impulses at $t = 0$
- $s = \sigma + j\omega$ is a complex variable
- s has units of inverse seconds (s^{-1})
- Known as the **one-sided** (unilateral) Laplace transform
- There is also a **two-sided** (bilateral) version:
  $X(s) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-st}\,\mathrm{d}t$
- We will only work with the one-sided version
  - + Easier to obtain the transient response
  - + Consistent with common practice
  - − Ignores $x(t)$ for $t < 0$

Laplace Transform Convergence

- The Laplace transform does not converge to a finite value for all signals and all values of s
- Does converge for all signals we will be interested in
  - Sinusoids
  - $\mathrm{e}^{-at}u(t)$ for any real $|a| < \infty$
  - $\delta(t)$
- The values of s for which the Laplace transform converges is called the **region of convergence (ROC)**
- Will not discuss in detail this term, but may see this in other classes on linear systems
- See *Signals and Systems* chapter for more information

Example 1: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = u(t)$. What is the region of convergence? What is the transform of $x(t) = 1$?

Example 2: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = \mathrm{e}^{-at}u(t)$. What is the region of convergence? What is the transform of $x(t) = \mathrm{e}^{-at}$?

Example 3: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = \delta(t)$. What is the region of convergence?

Example 4: Laplace Transform of $x(t)$

Find the Laplace transform of $x(t) = \cos(\omega t)u(t)$. What is the region of convergence? What is the Laplace transform of $\cos(\omega t)$?

Example 4: Workspace

Laplace Transform Properties

- The Laplace transform has many important properties
- We need to know these for at least three reasons
  - Improves our understanding of the transform
  - Enables us to find the transform more easily
  - Enables us to find the inverse transform more easily
- Will use the following notation for Laplace transform pairs

$$\begin{aligned} x(t)\,u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} X(s) \\ X(s) &= \mathcal{L}\{x(t)\} \\ x(t)\,u(t) &= \mathcal{L}^{-1}\{X(s)\} \end{aligned}$$

Linearity

$$\begin{aligned} X_1(s) &= \mathcal{L}\{x_1(t)\} \\ X_2(s) &= \mathcal{L}\{x_2(t)\} \end{aligned}$$

then you should be able to show that

$$[a_1 x_1(t) + a_2 x_2(t)]\, u(t) \overset{\mathcal{L}}{\Longleftrightarrow} a_1 X_1(s) + a_2 X_2(s)$$

Example: Find the Laplace transform of $x(t) = 5\delta(t) - 2\cos 5t$.

$$\begin{aligned} \mathcal{L}\{\delta(t)\} &= 1 \\ \mathcal{L}\{\cos \omega t\} &= \frac{s}{s^2 + \omega^2} \\ \mathcal{L}\{5\delta(t) - 2\cos 5t\} &= 5(1) - 2\left(\frac{s}{s^2 + 5^2}\right) \\ &= 5 - \frac{2s}{s^2 + 25} \end{aligned}$$

Scaling

Given $X(s) = \mathcal{L}\{x(t)\}$, what is $\mathcal{L}\{x(at)\}$ for $a > 0$?

$$\mathcal{L}\{x(at)\} = \int_{0^-}^{\infty} x(at)\mathrm{e}^{-st}\,\mathrm{d}t$$

$$\tau = at \qquad\qquad t = \frac{\tau}{a}$$

$$\mathrm{d}\tau = a\,\mathrm{d}t \qquad\qquad \mathrm{d}t = \frac{1}{a}\,\mathrm{d}\tau$$

$$\begin{aligned}\mathcal{L}\{x(at)\} &= \frac{1}{a}\int_{0^-}^{\infty} x(\tau)\mathrm{e}^{-s\frac{\tau}{a}}\,\mathrm{d}\tau \\ &= \frac{1}{a}\int_{0^-}^{\infty} x(\tau)\mathrm{e}^{-\left(\frac{s}{a}\right)\tau}\,\mathrm{d}\tau\end{aligned}$$

$$x(at)\,u(t) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{a}X\left(\frac{s}{a}\right)$$

Translation in Time

Given $X(s) = \mathcal{L}\{x(t)\}$, what is $\mathcal{L}\{x(t-t_0)\,u(t-t_0)\}$ for $t_0 > 0$?

$$
\begin{aligned}
\mathcal{L}\{x(t-t_0)u(t-t_0)\} &= \int_{0^-}^{\infty} x(t-t_0)\,u(t-t_0)\mathrm{e}^{-st}\,\mathrm{d}t \\
\tau &= t - t_0 \\
\mathrm{d}\tau &= \mathrm{d}t \\
t &= \tau + t_0 \\
\mathcal{L}\{x(t-t_0)\,u(t-t_0)\} &= \int_{-t_0}^{\infty} x(\tau)\,u(\tau)\mathrm{e}^{-s(\tau+t_0)}\,\mathrm{d}\tau \\
&= \int_{0^-}^{\infty} x(\tau)\,u(\tau)\mathrm{e}^{-s(\tau+t_0)}\,\mathrm{d}\tau \\
&= \mathrm{e}^{-st_0}\int_{0^-}^{\infty} x(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau \\
x(t-t_0)\,u(t-t_0) &\overset{\mathcal{L}}{\Longleftrightarrow} \mathrm{e}^{-st_0}X(s)
\end{aligned}
$$

Translation in Frequency

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the inverse Laplace transform of $X(s+s_0)$?

$$
\begin{aligned}
X(s+s_0) &= \int_{0^-}^{\infty} x(t)\,\mathrm{e}^{-(s+s_0)t}\,\mathrm{d}t \\
&= \int_{0^-}^{\infty} \left(x(t)\,\mathrm{e}^{-s_0 t}\right)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \mathcal{L}\left\{\mathrm{e}^{-s_0 t}x(t)\right\} \\
\mathcal{L}^{-1}\{X(s+s_0)\} &= \mathrm{e}^{-s_0 t}x(t)\,u(t) \\
\mathrm{e}^{-s_0 t}x(t)\,u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} X(s+s_0)
\end{aligned}
$$

Time Differentiation

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the Laplace transform of $\dot{x}(t)$?

$$\mathcal{L}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} = \int_{0^-}^{\infty} \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{e}^{-st}\,\mathrm{d}t$$

$$u = \mathrm{e}^{-st} \qquad \mathrm{d}u = -s\mathrm{e}^{-st}\,\mathrm{d}t$$

$$\mathrm{d}v = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\mathrm{d}t \qquad v = x(t)$$

$$\begin{aligned}
\mathcal{L}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} &= \int_{0^-}^{\infty} u\,\mathrm{d}v = \left. uv\right|_{0^-}^{\infty} - \int_{0^-}^{\infty} v\,\mathrm{d}u \\
&= \left.\mathrm{e}^{-st}x(t)\right|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)\left(-s\mathrm{e}^{-st}\right)\,\mathrm{d}t \\
&= \left(0 - x(0^-)\right) + s\int_{0^-}^{\infty} x(t)\,\mathrm{e}^{-st}\,\mathrm{d}t
\end{aligned}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}\,u(t) \stackrel{\mathcal{L}}{\Longleftrightarrow} sX(s) - x(0^-)$$

Time Differentiation Continued

In general

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\, u(t) \quad \overset{\mathcal{L}}{\Longleftrightarrow} \quad s^n X(s) - s^{n-1} x(0^-) - \cdots - s^0 \left. \frac{\mathrm{d}^{n-1} x(t)}{\mathrm{d}t^{n-1}} \right|_{t=0^-}$$

If all of the initial conditions are zero,

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t}\, u(t) \quad \overset{\mathcal{L}}{\Longleftrightarrow} \quad s^n X(s)$$

Time Integration

Given $X(s) = \mathcal{L}\{x(t)\}$, what is the Laplace transform of $\int_{0^-}^{t} x(\tau)\,\mathrm{d}\tau$?

$$\mathcal{L}\left\{\int_0^t x(\tau)\,\mathrm{d}\tau\right\} = \int_{0^-}^{\infty}\left(\int_0^t x(\tau)\,\mathrm{d}\tau\right)\mathrm{e}^{-st}\,\mathrm{d}t$$

$$u = \int_0^t x(\tau)\,\mathrm{d}\tau \qquad \mathrm{d}u = x(t)\,\mathrm{d}t$$

$$\mathrm{d}v = \mathrm{e}^{-st}\mathrm{d}t \qquad v = \frac{-1}{s}\mathrm{e}^{-st}$$

$$\begin{aligned}\mathcal{L}\left\{\int_0^t x(\tau)\,\mathrm{d}\tau\right\} &= \left(\int_0^t x(\tau)\,\mathrm{d}\tau\right)\frac{-1}{s}\mathrm{e}^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty}\frac{-1}{s}\mathrm{e}^{-st}x(t)\,\mathrm{d}t \\ &= (0-0) + \frac{1}{s}\int_{0^-}^{\infty} x(t)\mathrm{e}^{-st}\,\mathrm{d}t\end{aligned}$$

$$\int_0^t x(\tau)\,\mathrm{d}\tau \quad \overset{\mathcal{L}}{\Longleftrightarrow} \quad \frac{1}{s}X(s)$$

Other Properties

- Other properties of the Laplace transform are derived in the text
- See Table 15.1 (page 687) of the electric circuits text
- Common Laplace transform pairs are listed in Table 15.2 (Page 687)
- You should put copies of these tables on your notes that you bring to the exams

Example 5: Laplace Transform Properties

Find the Laplace transform of $t\,u(t)$.

Example 6: Laplace Transform Properties

Find the Laplace transform of

$$\dot{r}(t) \triangleq \frac{\mathrm{d}r(t)}{\mathrm{d}t}$$

Example 7: Laplace Transform Properties

Find the Laplace transform of $\mathrm{e}^{-at}\cos(\omega t)\,u(t)$. Hint: $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$

Inverse Laplace Transform Overview

$$\mathcal{L}^{-1}\left\{X(s)\right\} = \frac{1}{j2\pi}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)\,\mathrm{e}^{st}\,\mathrm{d}s$$

- The inverse Laplace transform is given above
- σ_1 is such that the integral is taken over a line in the region of convergence
- Very difficult to apply directly
- We will use a different approach
- Convert $X(s)$ to a form such that we can easily find the inverse

Inverse Laplace Transform Example

$$X(s) = \frac{s+8}{s(s+2)} = \frac{4}{s} - \frac{3}{s+2}$$

Since we know

$$\begin{aligned} u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{s} \\ \mathrm{e}^{-at}u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{s+a} \\ a_1 x_1(t)u(t) + a_2 x_2(t)u(t) &\overset{\mathcal{L}}{\Longleftrightarrow} a_1 X_1(s) + a_2 X_2(s) \end{aligned}$$

we know that the inverse Laplace transform of $X(s)$ is

$$\mathcal{L}^{-1}\left\{X(s)\right\} = 4u(t) - 3\mathrm{e}^{-2t}u(t)$$

Partial Fraction Expansion

A critical step in the previous example was finding following equation:

$$X(s) = \frac{s+8}{s(s+2)} = \frac{4}{s} - \frac{3}{s+2}$$

- In general, this can be done by partial fraction expansion
- In practice, we can do this using your calculators, MATLAB, or a similar tool

PFE: Overview

In general, most functions will have a general form

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{N} a_k s^k}{\sum_{k=0}^{M} b_k s^k}$$

- $N(s)$ and $D(s)$ are a polynomials in s.
- The roots of $N(s) = 0$ are called **zeros** of $X(s)$
- The roots of $D(s) = 0$ are called **poles** of $X(s)$
- To find $x(t) = \mathcal{L}^{-1}\{X(s)\}$, we need to
  1. Find the poles of $X(s)$
  2. Apply partial fraction expansion (via MATLAB)
  3. Find the inverse of each term by table lookup

PFE: Distinct Poles

$$\begin{aligned} X(s) &= \frac{N(s)}{D(s)} \\ &= \frac{N(s)}{(s+p_1)(s+p_2)\dots(s+p_n)} \\ &= \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \dots + \frac{k_n}{s+p_n} \\ k_i &= (s+p_i)X(s)\big|_{s=-p_i} \end{aligned}$$

- A distinct pole is a unique root of $D(s) = 0$
- The coefficients k_i are called the **residues** of $X(s)$
- To find k_i multiply both sides by $(s + p_i)$ and evaluate at $s = -p_i$

Example 8: Partial Fraction Expansion

Given $X(s) = \mathcal{L}\{x(t)\}$, find $x(t)$.

$$X(s) = \frac{5s + 29}{s^3 + 8s^2 + 19s + 12}$$

$\gg$ [r,p,k] = residue([5 29],[1 8 19 12])

r = 3.0000, -7.0000, 4.0000

p = -4.0000, -3.0000, -1.0000

k = []

Example 8: Workspace

PFE: Distinct Complex Poles Method 1

$$
\begin{aligned}
X(s) &= \frac{N(s)}{D(s)} \\
&= \frac{N(s)}{(s+\alpha-j\beta)(s+\alpha+j\beta)} \\
&= \frac{k_1}{s+\alpha-j\beta} + \frac{k_1^*}{s+\alpha+j\beta} \\
k_1 &= (s+\alpha-j\beta)X(s)\big|_{s=-\alpha+j\beta}
\end{aligned}
$$

- There are two methods for handling complex poles
- Often the residues of $X(s)$ will be complex
- In this case, the complex roots of $X(s)$ will be in complex conjugate pairs
- The residues will also be complex conjugate pairs

PFE: Distinct Complex Poles Method 2

$$
\begin{aligned}
X(s) &= \frac{N(s)}{D(s)} \\
&= \frac{k_1 s + k_2}{s^2 + as + b} + R(s) \\
&= \frac{k_1 s + k_1\alpha + k_2 - k_1\alpha}{(s+\alpha)^2 + \beta^2} + R(s) \\
&= \frac{k_1(s+\alpha) + k_2 - k_1\alpha}{(s+\alpha)^2 + \beta^2} + R(s) \\
&= \frac{c_1(s+\alpha)}{(s+\alpha)^2 + \beta^2} + \frac{c_2\beta}{(s+\alpha)^2 + \beta^2} + R(s) \\
c_1 &= k_1 \\
c_2 &= \frac{k_2 - k_1\alpha}{\beta} \\
x(t) &= c_1 \mathrm{e}^{-\alpha t}\cos(\beta t)u(t) + c_2\mathrm{e}^{-\alpha t}\sin(\beta t)u(t) + \mathcal{L}^{-1}\left\{R(s)\right\}
\end{aligned}
$$

Example 9: Distinct Complex Poles

Given $X(s) = \mathcal{L}\left\{x(t)\right\}$, find $x(t)$ using both methods of handling complex poles.

$$X(s) = \frac{2}{s^3 + 2s^2 + 2s}$$

$\gg$ [r,p,k] = residue([2],[1 2 2 0])

r = -0.5000 + 0.5000i, -0.5000 - 0.5000i, 1.0000

p = -1.0000 + 1.0000i, -1.0000 - 1.0000i, 0

k = []

Example 9: Workspace

PFE: Useful Transforms

$$\frac{k}{s+a} \Leftrightarrow k\mathrm{e}^{-at}u(t)$$

$$\frac{k}{(s+a)^2} \Leftrightarrow kt\mathrm{e}^{-at}u(t)$$

$$\frac{k}{s+\alpha-j\beta} + \frac{k^*}{s+\alpha+j\beta} \Leftrightarrow 2|k|\mathrm{e}^{-\alpha t}\cos(\beta t+\theta_k)u(t)$$

$$\frac{k}{(s+\alpha-j\beta)^2} + \frac{k^*}{(s+\alpha+j\beta)^2} \Leftrightarrow 2t|k|\mathrm{e}^{-\alpha t}\cos(\beta t+\theta_k)u(t)$$

where $k = |k|\angle\theta_k$

- You solve for k just as for a single pole (residue)
- Can also complete the square, as described in the textbooks

PFE: Repeated Poles

$$
\begin{aligned}
X(s) &= \frac{N(s)}{D(s)} \\
&= \frac{N(s)}{(s+p)^n} \\
&= \frac{k_1}{s+p} + \frac{k_2}{(s+p)^2} + \cdots + \frac{k_n}{(s+p)^n} \\
k_{n-m} &= \frac{1}{m!}\frac{\mathrm{d}^{(m)}}{\mathrm{d}s^m} \left.(s+p)^n X(s)\right|_{s=-p} \\
\mathcal{L}^{-1}\left\{\frac{1}{(s+p)^n}\right\} &= \frac{1}{(n-1)!} t^{n-1} \mathrm{e}^{-pt} u(t)
\end{aligned}
$$

- You can apply the equations above to handle repeated poles
- The **algebraic method** is usually easier

Example 10: Repeated Poles

Given $X(s) = \mathcal{L}\{x(t)\}$, find $x(t)$.

$$X(s) = \frac{s-2}{s(s+1)^3}$$

$\gg$ [r,p,k] = residue([1 -2],poly([0 -1 -1 -1]))

r = 2.0000, 2.0000, 3.0000, -2.0000

p = -1.0000, -1.0000, -1.0000, 0

k = []

Example 10: Workspace

PFE: Improper Rational Functions

$$X(s) = \frac{N(s)}{D(s)} = A(s) + \frac{B(s)}{C(s)}$$

- If $X(s)$ is an improper rational function, you must convert it to an expression that contains a proper rational function
- Will not explain conversion
- Key point: if the order of $N(s)$ is greater than or equal to the order of $D(s)$, you cannot apply PFE directly
- MATLAB's residue will find the coefficients of $A(s)$ as part of the partial fraction expansion

Solving Ordinary Differential Equations

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = x(t)$$

$$\sum_{k=0}^{N} a_k s^k Y(s) - \sum_{k=0}^{N} \sum_{\ell=1}^{k} s^{k-\ell} y^{(\ell-1)}(0^-) = X(s)$$

$$Y(s) = \frac{X(s) + \sum_{k=0}^{N} \sum_{\ell=1}^{k} s^{k-\ell} y^{(\ell-1)}(0^-)}{\sum_{k=0}^{N} a_k s^k}$$

If $\mathcal{L}\left\{x(t)\right\}$ is a rational function of s, then the linear ordinary differential equation shown above can be solved (more easily) using the Laplace Transform

Example 11: Solving ODE's

Solve the following ODE for $y(t)$ given that $y(0^-) = \dot{y}(0^-) = 0$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = 20\cos(2t)u(t) + \tfrac{1}{2}\sin(2t)u(t)$$

Hint:

$\gg$ [r,p,k] = residue([20 1],[conv([1 3 2],[1 0 4])])

r = 4.8750, -0.5375 - 1.4875i, -0.5375 + 1.4875i, -3.8000

p = -2.0000, -0.0000 + 2.0000i, -0.0000 - 2.0000i, -1.0000

k = []

$\gg$ [abs(r) angle(r)*180/pi]

ans = 4.8750 0, 1.5816 -109.8670, 1.5816 109.8670, 3.8000 180.0000,

Example 11: Workspace 1

Example 11: Workspace 2

Summary

- The Laplace transform can be used to solve ordinary differential equations
- This includes circuits and many other linear time-invariant systems
- We use the one-sided Laplace transform
- The inverse of this transform is always 0 for $t < 0$
- We usually solve for the inverse by using known transform pairs and the properties of the transform
- This topic is covered in both books