Laplace Transforms Circuit Analysis

- Passive element equivalents
- Review of ECE 221 methods in $s$ domain
- Many examples
Example 1: Circuit Analysis

We can use the Laplace transform for circuit analysis if we can define the circuit behavior in terms of a linear ODE.

For example, solve for \( v(t) \). Check your answer using the initial and final value theorems and the methods discussed in Chapter 7.
Example 1: Workspace

Hint: \[ [r, p, k] = \text{residue}([-2e-3 2e3], [1 1e6 0]) \]

\[ r = -0.0040, 0.0020, \]

\[ p = -1000000, 0 \]

\[ k = [] \]
Example 1: Workspace
Laplace Transform Circuit Analysis Overview

• LPT is useful for circuit analysis because it transforms differential equations into an algebra problem

• Our approach will be similar to the phasor transform
  1. Solve for the initial conditions
     – Current flowing through each inductor
     – Voltage across each capacitor
  2. Transform all of the circuit elements to the $s$ domain
  3. Solve for the $s$ domain voltages and currents of interest
  4. Apply the inverse Laplace transform to find time domain expressions

• How do we know this will work?
Kirchhoff’s Laws

\[ \sum_{k=1}^{N} v_k(t) = 0 \quad \sum_{k=1}^{N} V_k(s) = 0 \]
\[ \sum_{k=1}^{M} i_k(t) = 0 \quad \sum_{k=1}^{M} I_k(s) = 0 \]

- Kirchhoff’s laws are the foundation of circuit analysis
  - KVL: The sum of voltages around a closed path is zero
  - KCL: The sum of currents entering a node is equal to the sum of currents leaving a node
- If Kirchhoff’s laws apply in the \( s \) domain, we can use the same techniques that you learned last term (ECE 221)
- Apply the LPT to both sides of the time domain expression for these laws
- The laws hold in the \( s \) domain
Defining \( s \) Domain Equations: Resistors

\[ i(t) \rightarrow \frac{R}{+ v(t) -} \quad I(s) \rightarrow \frac{R}{+ V(s) -} \]

\[ v(t) = R i(t) \quad \quad V(s) = R I(s) \]

- Generalization of Ohm’s Law
- As with KCL & KVL, the relationship is the same in the \( s \) domain as in the time domain
- Note that we used the linearity property of the LPT for both Ohm’s law and Kirchhoff’s laws
Defining $s$ Domain Equations: Inductors

\[ v(t) = L \frac{dI(t)}{dt} \]
\[ V(s) = L [sI(s) - I_0] \]
\[ V(s) = sLI(s) - LI_0 \]

Where $I_0 \triangleq i(0^-)$
Defining $s$ Domain Equations: Capacitors

\[
\begin{align*}
  i(t) &= \frac{C}{v(t)} + i(t) \\
  I(s) &= \frac{1}{sC} V(s) - I(s) \quad I(s) = \frac{1}{sC} V(s) - V_0 \\
  i(t) &= C \frac{dv(t)}{dt} \\
  I(s) &= C [sV(s) - V_0] \\
  I(s) &= sCV(s) - CV_0 \\
  v(t) &= \frac{1}{C} \int_0^t i(\tau) d\tau + V_0 \\
  V(s) &= \frac{1}{C} \left[ \frac{1}{s} I(s) \right] + \frac{1}{s} V_0 \\
  V(s) &= \frac{1}{sC} I(s) + \frac{V_0}{s}
\end{align*}
\]

Where $V_0 \triangleq v(0^-)$
**s Domain Impedance and Admittance**

Impedance: \[ Z(s) = \frac{V(s)}{I(s)} \]

Admittance: \[ Y(s) = \frac{I(s)}{V(s)} \]

- The *s* domain **impedance** of a circuit element is defined for zero initial conditions.
- This is also true for the *s* domain admittance.
- We will see that circuit *s* domain circuit analysis is easier when we can assume *zero initial conditions*.
### $s$ Domain Circuit Element Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>$V(s) = RI(s)$</th>
<th>$V = RI$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$V(s) = sLI(s)$</td>
<td>$V = sLI$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$V(s) = \frac{1}{sC} I(s)$</td>
<td>$V = \frac{1}{sC} I$</td>
</tr>
</tbody>
</table>

- All of these are in the form $V(s) = ZI(s)$
- Note similarity to phasor transform
- Identical if $s = j\omega$
- Will discuss further later
- Equations only hold for zero initial conditions
Example 2: Circuit Analysis

Solve for $v(t)$ using $s$-domain circuit analysis.
Example 2: Workspace
Example 3: Circuit Analysis

Given $v_o(0) = 0$, solve for $v_o(t)$ for $t \geq 0$. 

\[ v_o(t) = \sin(1000t) \]
Example 3: Workspace

Hint: \[ r, p, k = \text{residue}([1e6], \text{conv}([1 0 1e6],[1 1e3])) \]

\[ r = [0.5000, -0.2500 - 0.2500i, -0.2500 + 0.2500i] \]

\[ p = 1.0e+003 \times [-1.0000, 0.0000 + 1.0000i, 0.0000 - 1.0000i] \]

\[ k = [] \]

\[ \gg \text{abs}(r) \text{ angle}(r) \times 180/\pi \]

\[ \text{ans} = [0.5000 \text{ } 0, 0.3536 \text{ } -135.0000, 0.3536 \text{ } 135.0000] \]
Example 3: Workspace
Example 3: Plot of Results

$\text{Time (ms)}$

$\text{v}_o(t) (\text{V})$

-\frac{8}{10}$ \quad \frac{8}{10}$

-\frac{2}{10}$ \quad \frac{2}{10}$

-\frac{4}{10}$ \quad \frac{4}{10}$

-\frac{6}{10}$ \quad \frac{6}{10}$

-\frac{8}{10}$ \quad \frac{8}{10}$

$\text{Total}$

$\text{Transient}$

$\text{Steady State}$
Example 4: Circuit Analysis

Solve for $v(t)$. 
Example 4: Workspace

Hint: \[ r, p, k = \text{residue}([1e-3 20 0],[1 21.25e3 10e3]) \]

\[ r = [-1.2496, -0.0004] \]

\[ p = [-21250, -0.4706] \]

\[ k = [0.0010] \]
Example 4: Workspace
Example 5: Parallel RLC Circuits

Find an expression for $V(s)$. Assume zero initial conditions.
Example 6: Circuit Analysis

Given $v(0) = 0 \text{ V}$ and the current through the inductor is $i_L(0^-) = -12.25 \text{ mA}$, solve for $v(t)$. 
Example 6: Workspace

Hint: \[ r, p, k \] = residue([98e3],[1 400 1e6])

\[ r = [0 -50.0104i,0 +50.0104i] \]

\[ p = 1.0e+002 \times [-2.0000 + 9.7980i,-2.0000 - 9.7980i] \]

\[ k = [] \]

\[ \text{abs}(r) \text{ angle}(r) \times 180/\pi \]

\[ \text{ans} = [50.0104 -90.0000, 50.0104 90.0000] \]
Example 6: Workspace
Example 6: Plot of $v(t)$
Example 6: MATLAB Code

t = 0:0.01e-3:40e-3;
v = 50*exp(-200*t).*sin(979.8*t);
t = t*1000;
h = plot(t,v,'b');
set(h,'LineWidth',1.2);
xlim([0 max(t)]);
ylim([-23 40]);
box off;
xlabel('Time (ms)');
ylabel('(volts)');
title('');
Example 7: Series RLC Circuits

Find an expression for $V_R(s)$, $V_L(s)$, and $V_C(s)$. Assume zero initial conditions.
Example 7: Workspace
Example 8: Circuit Analysis

There is no energy stored in the circuit at $t = 0$. Solve for $v_2(t)$. 
Example 8: Workspace

Hint: \( [r, p, k] = \text{residue}([320e3],[1 5e3 0]) \)

\( r = [-64, 64] \)

\( p = [-5000, 0] \)

\( k = [] \)
Example 8: Workspace
Example 9: Circuit Analysis

Solve for $V_2(s)$. Assume zero initial conditions.
Example 9: Workspace
Example 9: Workspace
Example 10: Circuit Analysis

Find the Thevenin equivalent of the circuit above. Assume that the capacitor is initially uncharged.
Example 10: Workspace
Example 11: Circuit Analysis

Find an expression for $v_o(t)$ given that $v(t) = e^{-\alpha t}u(t)$ and $i_L(0) = I_0$ mA.
Example 11: Workspace
Example 11: Workspace
Find an expression for \( V_o(s) \) in terms of \( V_s(s) \). Assume there is no energy stored in the circuit initially. What is \( v_o(t) \) if \( v_s(t) = u(t) \)?
Example 12: Workspace

Hint: \[ r,p,k] = \text{residue}([-0.2\ 0],\text{conv}([1\ 2e3],[1\ 1e3])) \]

\[ r = [-0.4000, 0.2000] \]

\[ p = [-2000, -1000] \]

\[ k = [] \]
Example 12: Workspace
Example 13: Circuit Analysis

Find an expression for $V_o(s)$ in terms of $V_s(s)$. Assume there is no energy stored in the circuit initially.
Example 13: Workspace
Example 13: Workspace
Find an expression for $V_o(s)$ in terms of $V_s(s)$. Assume there is no energy stored in the circuit initially.
Example 14: Workspace