Overview of Bode Plots

- Transfer function review
- Piece-wise linear approximations
- First-order terms
- Second-order terms (complex poles & zeros)

Transfer Function Review

\[
x(t) \rightarrow H(s) \rightarrow y(t)
\]

Recall that if \( H(s) \) is known and

\[
x(t) = A \cos(\omega t + \phi),
\]

then we can find the steady-state solution for \( y(t) \):

\[
y_{ss}(t) = A|H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))
\]

Bode Plots

\[
x(t) \rightarrow H(s) \rightarrow y(t)
\]

- Bode plots are standard method of plotting the magnitude and phase of \( H(s) \)
- Both plots use a logarithmic scale for the \( x \)-axis
- Frequency is in units of radians/second (rad/s)
- The phase is plotted on a linear scale in degrees
- Magnitude is plotted on a linear scale in decibels

\[
H_{\text{dB}}(j\omega) \triangleq 20 \log_{10} |H(j\omega)|
\]

Decibel Scales

It is important to become adept at translating between amplitude, \( |H(j\omega)| \), and decibels, \( H_{\text{dB}}(j\omega) \).

| Amplitude (\(|H(j\omega)|\)) | Decibels (\(20 \log_{10} |H(j\omega)|\)) |
|-----------------------------|---------------------------------|
| 1                           | \(20 \log_{10} 1\) =            |
| 10                          | \(20 \log_{10} 10\) =           |
| 100                         | \(20 \log_{10} 100\) =          |
| 1000                        | \(20 \log_{10} 1000\) =         |
| 0.1                         | \(20 \log_{10} 0.1\) =          |
| 0.01                        | \(20 \log_{10} 0.01\) =         |
| 0.001                       | \(20 \log_{10} 0.001\) =        |
| \(\frac{1}{2}\)            | \(20 \log_{10} \frac{1}{2}\) =  |
| 2                           | \(20 \log_{10} 2\) =            |
| \(\sqrt{\frac{1}{2}}\)    | \(20 \log_{10} \sqrt{\frac{1}{2}}\) =  |
Example 1: Bode Plots

1. Find the transfer function of the circuit shown above.
2. Generate the bode plot.

Example 1: MATLAB Code

```matlab
w = logspace(1,5,500);
H = -50e3./(j*w + 5e3);
subplot(2,1,1);
h = semilogx(w,20*log10(abs(H)));
set(h,'LineWidth',1.4);ylabel('|H(j\omega)| (dB)');
title('Active Lowpass RC Filter');
set(gca,'Box','Off');grid on;
set(gca,'YLim',[-5 25]);
subplot(2,1,2);
h = semilogx(w,angle(H)*180/pi);
set(h,'LineWidth',1.4);ylabel('\angle H(j\omega) (degrees)');
set(gca,'Box','Off');grid on;
set(gca,'YLim',[85 185]);
xlabel('Frequency (rad/s)');
```

Example 1: Workspace
1. Find the transfer function of the circuit shown above.
2. Generate the bode plot.
Bode Plot Approximations

- Until recently (late 1980's) bode plots were drawn by hand
- There were many rules-of-thumb, tables, and template plots to help
- Today engineers primarily use MATLAB, or the equivalent
- Why discuss the old method of plotting by hand?
  - It is still important to understand how the poles, zeros, and gain influence the Bode plot
  - These ideas are used for transfer function synthesis, analog circuit design, and control systems
- We will discuss simplified methods of generating Bode plots
- Based on asymptotic approximations

Alternate Transfer Function Expressions

There are many equivalent expressions for transfer functions.

\[
H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{b_m}{a_n} s^{\pm \ell} \left( s - z_1 \right) \left( s - z_2 \right) \cdots \left( s - z_m \right) \left( s - p_1 \right) \left( s - p_2 \right) \cdots \left( s - p_n \right) = k s^{\pm \ell} \left( 1 - \frac{s}{z_1} \right) \left( 1 - \frac{s}{z_2} \right) \cdots \left( 1 - \frac{s}{z_m} \right) \left( 1 - \frac{s}{p_1} \right) \left( 1 - \frac{s}{p_2} \right) \cdots \left( 1 - \frac{s}{p_n} \right)
\]

- This last expression is called standard form
- The first step in making bode plots is to convert \( H(s) \) to standard form

Magnitude Components

Consider the expression for the transfer function magnitude:

\[
|H_{dB}(j\omega)| = 20 \log_{10} |H(j\omega)|
\]

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Magnitude Components Comments

\[
|H_{dB}(\omega)| = 20 \log_{10} |k| \pm \ell 20 \log_{10} \omega + 20 \log_{10} \left| 1 - \frac{j\omega}{z_1} \right| + \cdots + 20 \log_{10} \left| 1 - \frac{j\omega}{z_m} \right|
\]

- Thus, \( |H_{dB}(\omega)| \) can be written as a sum of simple functions
- This is similar like using basis functions \( \{ \delta(t), u(t), \& r(t) \} \) to write an expression for a piecewise linear signal
- We will use this approach to generate our piecewise linear approximations of the bode plot
- Note that there are four types of components in this expression
  - Constant
  - Linear term
  - Zeros
  - Poles
Consider two limiting conditions for a term containing a zero, $20 \log_{10} |1 - \frac{j\omega}{z}|$.

First condition: $\omega \ll |z|$

$$\lim_{z \to 0} \frac{20 \log_{10} |1 - \frac{j\omega}{z}|}{|z|} = 0$$

Thus, if $\frac{\omega}{|z|} \ll 1$, then $20 \log_{10} |1 - \frac{j\omega}{z}| \approx 0$.

Second condition: $\omega \gg |z|$

$$\lim_{z \to \infty} \frac{20 \log_{10} |1 - \frac{j\omega}{z}|}{|z|} = 20 \log_{10} |\omega| - 20 \log_{10} |z|$$

Thus, if $\frac{\omega}{|z|} \gg 1$, then this term is linear (on a log scale) with a slope of $20$ dB per decade and an $x$-axis intercept at $\omega = |z|$.

The constant term, $20 \log_{10} |k|$, is a straight line on the Bode plot.

The linear term, $\pm \ell 20 \log_{10} |\omega|$, is a line on the magnitude plot with a slope equal to $\pm \ell 20$ dB per decade.

The $x$-axis intercept occurs at $\omega = 1$ rad/s.

Plot the bode magnitude plots for $H(s) = s$, $\frac{1}{s}$, $s^2$, $\frac{1}{s^2}$.
Magnitude Components: Real Poles

The approximation is least accurate at \( \omega = |p| \). The true magnitude is 3 dB less than the approximation at this corner frequency.

Magnitude Components: Real Zeros

The approximation is least accurate at \( \omega = |z| \). The true magnitude is 3 dB higher than the approximation at this corner frequency.

Complex Poles & Zeros

- Complex poles and zeros require special attention
- Will discuss later
- You will not be expected to plot approximations with complex poles or zeros on exams
- There are essentially 3 steps to generating piecewise linear approximations of bode plots
  1. Convert to standard form
  2. Plot the components
  3. Graphically add the components together
Example 3: Magnitude Components

Draw the piecewise approximation of the bode magnitude plot for

\[ H(s) = \frac{(s + 10)(s + 100)^2}{10s^2(s + 1000)} \]

Example 4: Magnitude Components

Draw the piecewise approximation of the bode magnitude plot for

\[ H(s) = \frac{10^{11}s(s + 100)}{(s + 10)(s + 1000)(s + 10,000)^2} \]
Phase Components

\[ H(j\omega) = k \frac{(j\omega)^{\ell} (1 - \frac{j\omega}{z_1}) \ldots (1 - \frac{j\omega}{z_m})}{(1 - \frac{j\omega}{p_1}) \ldots (1 - \frac{j\omega}{p_n})} \]

Each of these terms can be expressed in polar form: \( a + jb = Ae^{j\theta} \).

Note that \( (j\omega)^{\ell} = \omega^{\ell} e^{j\ell\pi/2} \).

\[ H(j\omega) = |k| e^{j\eta_\ell \pi} \frac{\prod N_i e^{j\theta_i} \ldots N_m e^{j\theta_m}}{D_1 e^{j\phi_1} \ldots D_n e^{j\phi_n}} \]

\[ = |k| \frac{\prod i \left|\omega^{\ell} N_1 \ldots N_m \right|}{D_1 \ldots D_n} \times \exp \left( j(\eta_\ell \pi + \ell \frac{\pi}{2} + \theta_1 + \ldots + \theta_n - \phi_1 - \ldots - \phi_n) \right) \]

where

\[ \eta_\ell = \begin{cases} 0 & k \geq 0 \\ 1 & k < 0 \end{cases} \]

\[ \angle H(j\omega) = \eta_\ell \pi + \ell \frac{\pi}{2} + \theta_1 + \ldots + \theta_n - \phi_1 - \ldots - \phi_n = \eta_\ell \pi + \ell \frac{\pi}{2} + \angle \left( 1 - \frac{j\omega}{z_1} \right) + \ldots + \angle \left( 1 - \frac{j\omega}{z_m} \right) - \angle \left( 1 - \frac{j\omega}{p_1} \right) - \ldots - \angle \left( 1 - \frac{j\omega}{p_n} \right) \]

- Thus the phase of \( H(j\omega) \) is also a linear sum of the phases due to each component.
- We will consider each of the four components in turn:
  - Constant
  - Linear term
  - Zeros
  - Poles
Phase Components: Real Zeros

Consider three limiting conditions for a term containing a zero, $1 - \frac{j\omega}{z}$.

**First condition:** $\omega \ll |z|$

$$\lim_{\omega \to 0} \angle \left(1 - \frac{j\omega}{z}\right) = 0^\circ$$

Thus, if $\frac{\omega}{|z|} \ll 1$, then $\angle \left(1 - \frac{j\omega}{z}\right) \approx 0^\circ$.

**Second condition:** $\omega = |z|$

$$\angle \left(1 - \frac{j\omega}{z}\right)|_{\omega = |z|} = \angle \left(1 - j\eta_z\right) = -\eta_z 45^\circ$$

where $\eta_z = \text{sign}(z)$

**Third condition:** $\omega \gg |z|$

$$\lim_{z \to \infty} \angle \left(1 - \frac{j\omega}{z}\right) = \angle \left(-\frac{j\omega}{z}\right) = -\eta_z 90^\circ$$

Thus, if $\frac{\omega}{|z|} \gg 1$, then $\angle \left(1 - \frac{j\omega}{z}\right) \approx -\eta_z 90^\circ$.

The complex angle of the constant term, $k$, is either $0^\circ$ if $k > 0$ or $180^\circ$ if $k < 0$.

Phase Components: Linear Term

The linear term, $\angle (j\omega)^\ell = \angle j^\ell = \ell \times 90^\circ$, is a constant multiple of $90^\circ$. Plot the bode phase plots for $H(s) = s, \frac{1}{s}, s^2, \frac{1}{s^2}$.
Phase Components: Real Zeros in Right Plane

The approximation is least accurate at $\omega = 0.1|z|$ and $\omega = 10|z|$.

Phase Components: Real Zeros Continued 2

If the zero is in the right half plane (i.e. $\text{Re}\{z\} > 0$), then the phase approaches $-90^\circ$ asymptotically.

Plot the piecewise approximation of the term $\angle \left( 1 - \frac{j\omega}{z} \right)$. Assume that $z$ is in the right half plane.

Phase Components: Real Zeros in Left Plane

The approximation is least accurate at $\omega = 0.1|z|$ and $\omega = 10|z|$.

Phase Components: Real Poles

Real poles in the left half plane have the same phase as real zeros in the right half plane. We will only discuss poles in the left half plane because only these systems are stable.

First condition: $\omega \ll |p|$

$$\lim_{\frac{\omega}{p} \to 0} -\angle \left( 1 - \frac{j\omega}{p} \right) = -0^\circ$$

Second condition: $\omega = |p|$

$$-\angle \left( 1 - \frac{j\omega}{p} \right) \bigg|_{\omega = \text{sign}(p) \times j} = -\angle \left( 1 - \text{sign}(p) \times j \right) = -\angle (1 + j) = -45^\circ$$

Third condition: $\omega \gg |p|$

$$\lim_{\frac{\omega}{p} \to \infty} -\angle \left( 1 - \frac{j\omega}{p} \right) = -\angle \left( \frac{-j\omega}{p} \right) = -\angle j = -90^\circ$$
Example 5: Phase Components

\[ H(j\omega) = \frac{(s + 10)(s + 100)^2}{10s^2(s + 1000)} \]

Plot the piecewise approximation of the term \(-\angle\left( 1 - \frac{j\omega}{p} \right)\). Assume that \(p\) is in the left-hand plane (i.e. \(\text{Re}\{p\} < 0\)).

The approximation is least accurate at \(\omega = 0.1|p|\) and \(\omega = 10|p|\).
Example 6: Phase Components

\[
\angle H(j\omega) (\text{deg})
\]

Draw the piecewise approximation of the bode phase plot for

\[
H(s) = \frac{10^{11}s(s + 100)}{(s + 10)(s + 1000)(s + 10,000)^2}
\]

Example 7: Magnitude

\[
|H(j\omega)| (\text{dB})
\]

Plot the magnitude of

\[
H(s) = 10 \frac{s + 10}{s + 1000}
\]

Example 7: Phase

\[
\angle H(j\omega) (\text{deg})
\]

Plot the phase of

\[
H(s) = 10 \frac{s + 10}{s + 1000}
\]
Example 8: Phase

\[ H(j\omega) = -10 \frac{s - 10}{s + 1000} \]

Plot the phase of \( H(s) = -10 \frac{s - 10}{s + 1000} \)

Example 7: Solution

\begin{align*}
H(s) &= \frac{10}{s + 10} \\
H(j\omega) &= \frac{10}{\sqrt{1 + 100\omega^2}} \\
\angle H(j\omega) &= -\arctan(10\omega) \\
\end{align*}

Example 8: Magnitude

\[ |H(j\omega)| (dB) = 20 \log_{10} \left| \frac{10}{\sqrt{1 + 100\omega^2}} \right| \]

Example 8: Phase

\[ \angle H(j\omega) (deg) = -\arctan(10\omega) \]

Example 8: Solution

\begin{align*}
H(s) &= \frac{10}{s + 10} \\
H(j\omega) &= \frac{10}{\sqrt{1 + 100\omega^2}} \\
\angle H(j\omega) &= -\arctan(10\omega) \\
\end{align*}
Example 9: Circuit Example

\[ H(s) = \frac{\frac{R}{L} s}{s^2 + \frac{R}{L} s + \frac{1}{LC}} \]

Draw the straight-line approximations of the transfer function for the circuit shown above. Hint: Recall from one of the Transfer Functions Examples.
### Complex Poles

Complex poles can be expressed in the following form:

\[ C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{1 + 2\zeta \frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2} \]

- \( \omega_n \) is called the **undamped natural frequency**
- \( \zeta \) (zeta) is called the **damping ratio**
- The poles are \( p_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1}) \omega_n \)
- If \( \zeta \geq 1 \), the poles are real
- If \( 0 < \zeta < 1 \), the poles are complex
- If \( \zeta = 0 \), the poles are imaginary: \( p_{1,2} = \pm j\omega_n \)
- If \( \zeta < 0 \), the poles are in the right half plane (\( \text{Re}\{p\} > 0 \)) and the system is unstable

### Complex Poles Magnitude

\[
20 \log_{10} |C(j\omega)| = 20 \log_{10} \left| 1 - \left(\frac{\omega}{\omega_n}\right)^2 + \frac{j\omega}{Q\omega_n}\right|^{-1}
\]

\[
= -20 \log_{10} \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{\omega}{Q\omega_n}\right)^2}
\]

For \( \omega \ll \omega_n \),

\[ 20 \log_{10} |C(j\omega)| \approx -20 \log_{10} |1| = 0 \text{ dB} \]

For \( \omega \gg \omega_n \),

\[ 20 \log_{10} |C(j\omega)| \approx -20 \log_{10} \frac{\omega^2}{\omega_n^2} = -40 \log_{10} \frac{\omega}{\omega_n} \text{ dB} \]

At these extremes, the behavior is identical to two real poles.
**Complex Poles Phase**

\[ \angle C(j\omega) = \angle \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + \frac{j\omega}{Q\omega_n}} \]

For \( \omega \ll \omega_n \),
\[ \angle C(j\omega) \approx \angle 1 = 0^\circ \]

For \( \omega \gg \omega_n \),
\[ \angle C(j\omega) \approx \angle \frac{1}{-\frac{\omega}{Q\omega_n}} = \angle -\frac{\omega\omega_n}{\omega} = \angle -1 = -180^\circ \]

For \( \omega = \omega_n \),
\[ \angle C(j\omega) \approx \angle \frac{1}{Qj} = \angle -j = -90^\circ \]

At the extremes, the behavior is identical to two real poles. At other values of \( \omega \) near \( \omega_n \), the behavior is more complicated.

**Complex Zeros**

- **Left half plane:**
  - Inverted magnitude of complex poles
  - Inverted phase of complex poles
- **Right half plane:**
  - Inverted magnitude of complex poles
  - Same phase of complex poles
- This is the same relationship real zeros had to real poles
Complex Poles Maximum

What is the frequency at which \(|C(j\omega)|\) is maximized?

\[
C(j\omega) = \frac{1}{1 + \left(\frac{j\omega}{Q\omega_n}\right)^2 + \left(\frac{j\omega}{Q\omega_n}\right)^2}
\]

\[
|C(j\omega)| = \frac{1}{\sqrt{1 - \frac{\omega^2}{\omega_n^2} + \left(\frac{\omega}{Q\omega_n}\right)^2}}
\]

- For high values of \(Q\), the maximum of \(|C(j\omega)| > 1\)
- This is called **peaking**
- The largest \(Q\) before the onset of peaking is \(Q = \frac{1}{\sqrt{2}} \approx 0.707\)
- This curve is said to be maximally flat
- This is also called a **Butterworth response**
- In this case, \(|C(j\omega_n)| = -3\, \text{dB}\) and \(\omega_n\) is the cutoff frequency

If \(Q > 0.707\), the maximum magnitude and frequency are as follows:

\[
\omega_r = \omega_n \sqrt{1 - \frac{1}{2Q^2}} \quad |C(j\omega_r)| = \frac{Q}{1 - \frac{1}{4Q^2}}
\]

- \(\omega_r\) is called the **resonant frequency** or the **damped natural frequency**
- As \(Q \to \infty\), \(\omega_r \to \omega_n\)
- For sufficiently large \(Q\) (say \(Q > 5\))
  - \(\omega_r \approx \omega_n\)
  - \(|C(j\omega_r)| \approx Q\)
- Peaked responses are useful in the synthesis of high-order filters
- Complex zeros (in the left half plane) have the inverted magnitude and phase of complex poles
Complex Poles Example

\[ H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + \frac{R}{Q}s + \omega_n^2} \]

Generate the bode plot for the circuit shown above.

\[ \omega_n = \sqrt{\frac{1}{LC}} = 10 \text{ k rad/s} \]
\[ \zeta = \frac{R}{2L} \sqrt{\frac{1}{LC}} = \frac{R}{2} \sqrt{\frac{C}{L}} = R \times 0.001 \]
\[ Q = \frac{1}{\sqrt{LC}} \frac{L}{R} = \sqrt{\frac{L}{C}} \frac{R}{R} = 500 \]

- \( R = 5 \Omega \) \( \zeta = 0.005 \) \( Q = 100 \) Very Light Damping
- \( R = 50 \Omega \) \( \zeta = 0.05 \) \( Q = 10 \) Light Damping
- \( R = 707 \Omega \) \( \zeta = 1.41 \) \( Q = 0.707 \) Strong Damping
- \( R = 1 \text{ k} \Omega \) \( \zeta = 1 \) \( Q = 0.5 \) Critical Damping
- \( R = 5 \text{ k} \Omega \) \( \zeta = 5 \) \( Q = 0.1 \) Over Damping