Introduction: Steady-State Analysis

Consider a general linear circuit with some current or voltage of interest treated as the output.

If the input is a sinusoidal source (either voltage or current) applied at $t = 0$, then the output response can be divided into two components:

- The **steady-state response** is the part of the response that remains as $t \to \infty$.
- The **transient response** is the part of the response that approaches zero as $t \to \infty$.

Given a linear circuit with some current or voltage of interest treated as the output.

Consider a general linear circuit with some current or voltage of interest treated as the output.

- The **steady-state response** is the part of the response that remains as $t \to \infty$.
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Many examples:

- **Phasor analysis**
- Impedance combinations
- Kirchhoff's laws revised
- Circuit element defining equations revised
- Phasors
- Sinusoidal steady-state analysis

Example 1: Steady-State Analysis

Given $v_o(0) = 0$, solve for $v_o(t)$ for $t \geq 0$.

\[
\begin{align*}
v_o(t) &= \left(1/2 \right)e^{-t/0.001} + \left(1/\sqrt{2} \right)\sin(1000t - 45^\circ) \\
v_{\text{tr}}(t) &= \left(1/2 \right)e^{-t/0.001} \\
v_{\text{ss}}(t) &= \left(1/\sqrt{2} \right)\sin(1000t - 45^\circ)
\end{align*}
\]
Phasors

A \cos(\omega t + \theta) \iff A \angle \theta

For historical reasons, \( \omega \) will always have units of rads/s and \( \theta \) will always have units of degrees.

Phasor: a complex number that represents the amplitude and phase of a sinusoid.

You will need to learn how to manipulate complex numbers efficiently. The advanced scientific calculators should make this much easier.

We will use \( j = \sqrt{-1} \).

Why not use \( i \) like mathematicians?

Example: \( z = x + jy \)

Real operator example: \( \text{Re}\{z\} = x \)

Imaginary operator example: \( \text{Im}\{z\} = y \)

Complex Conjugate:

\[ \phi_{\text{conj}} = \{ \phi_{\text{real}} \} \]

Euler's identity:

\[ e^{j\phi} = \cos\phi + j\sin\phi \]

Re \{ e^{j\phi} \} = \cos\phi

Im \{ e^{j\phi} \} = \sin\phi

Complex Conjugate:

\[ z^* = x - jy \]

Euler's identity:

\[ e^{-j\phi} = \cos\phi - j\sin\phi \]

Re \{ e^{-j\phi} \} = \cos\phi

Im \{ e^{-j\phi} \} = -\sin\phi

Steady-State Analysis Comments

This chapter we will discuss circuits driven by sinusoidal sources exclusively.

We will only solve for the steady-state component of the total response for ECE 222. We will only solve for the total response for ECE 222. We will learn how to solve for the total response for non-sinusoidal sources exclusively.

This chapter we will discuss circuits driven by sinusoidal sources exclusively.

Example 1: Continued

\[ v_{tr}(t) + v_{ss}(t) = v_o(t) \]

Complex Numbers Review

Three representations of complex numbers:

\[ z = x + jy \]

\[ z = r \angle \phi \]

\[ z = re^{j\phi} \]

where

\[ r = \sqrt{x^2 + y^2} \]

\[ \phi = \text{angle } (x, y) \]

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Example 1: Continued
Find equivalent expressions for the following.

1. \( e^{-j90^\circ} \)  
2. \( 1 \angle 135^\circ \)  
3. \( 1 \angle 45^\circ \)  
4. \( e^{j270^\circ} \)  
5. \( 1 \angle 180^\circ \)  
6. \( 1 \angle -180^\circ \)  
7. \( 1 \angle 90^\circ \)
To use phasors for circuit analysis, we need to know how the laws of circuit analysis apply in the phasor domain.

- What are the defining equations for circuit elements?
- How do Kirchhoff’s laws apply?

To use phasors, we represent the amplitude and phase of steady-state sinusoidal circuit components by a single complex number.

A phasor transform represents the amplitude and phase of sinusoids by a single complex number.

\[
\begin{align*}
\phi V &= \phi \angle \theta \\
\cos(\phi + \theta) &= \cos \phi \cos \theta - \sin \phi \sin \theta
\end{align*}
\]
Phasor Transform: Summary

### Phasor Transform: Resistors

**Element Equation**  
\[ v(t) = Ri(t) \]

**Phasor Equation**  
\[ V = RI \]

- **Resistor**

### Phasor Transform: Inductors

**Element Equation**  
\[ v(t) = L \frac{di(t)}{dt} \]

**Phasor Equation**  
\[ V = j\omega LI \]

- **Inductor**

### Phasor Transform: Capacitors

**Element Equation**  
\[ i(t) = C \frac{dv(t)}{dt} \]

**Phasor Equation**  
\[ I = j\omega C V \]

- **Capacitor**

### Phasor Analysis: Impedance and Admittance

- In the phasor domain, there is a linear relationship between **I** and **V** for all three circuit elements:
  \[ V = ZI \]

- This is a generalization of Ohm’s law

- In the phasor domain, the constant coefficients **Z** and **Y** may be complex

- **Z** is called impedance (ohms - Ω)

- **Y** is called admittance (siemens - S)

### Phasor Transform: Capacitors

\[ i(t) = C \frac{dv(t)}{dt} = C (-\omega A \sin(\omega t + \phi)) = -\omega CA \cos(\omega t + \phi - 90^\circ) \]

### Phasor Transform: Resistors

\[ i(t) = Ae^{j\phi} \]

\[ v(t) = RA \cos(\omega t + \phi) \]

### Phasor Transform: Inductors

\[ I = A \cos(\omega t + \phi) \]

\[ V = R(i) = (i) \]

### Phasor Transform: Resistors

\[ I = A \cos(\omega t + \phi) \]

\[ V = R(i) = (i) \]

### Phasor Transform: Capacitors

\[ i(t) = C \frac{dv(t)}{dt} = C (-\omega A \sin(\omega t + \phi)) = -\omega CA \cos(\omega t + \phi - 90^\circ) \]

\[ V = Ae^{j\phi} \]

\[ I = -\omega CAe^{j(\phi - 90^\circ)} = -\omega CAe^{j\phi}e^{-j\pi/2} = j\omega C (Ae^{j\phi}) \]

\[ I = j\omega C V \]

\[ V = \frac{1}{j\omega C} I \]

### Phasor Transform: Resistors

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\[ I = j\omega C V \]

\[ V = \frac{1}{j\omega C} I \]
The same arguments can be applied to KCL. Recall that KCL states the sum of currents leaving (or entering) a node is equal to zero.

\[ 0 = i_1(t) + i_2(t) + \cdots + i_N(t) \]

The phasor equivalent of the above expression is:

\[ 0 = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2) + \cdots + A_N \cos(\omega t + \phi_N) \]

Thus, KCL applies in the phasor domain as well as the time domain.

\[ N I + \cdots + \bar{I} + I = 0 \]

What about Kirchhoff’s laws?

- \( B \) called conductance
- \( C \) called inductance
- \( X \) called reactance
- \( R \) called resistance

\[ Bf + C = X'f + Y = Z \]

Where is the equivalent of KVL in the phasor domain? Recall KVL:

\[ \text{The sum of voltages around a closed path is equal to zero.} \]

Phasor Transform: KVL

Phasor Analysis: Impedance and Admittance Continued

- In general, impedance and admittance will be complex numbers
- \( Z = R + jX \) and \( Y = G + jB \)
- \( R \) called resistance (\( \Omega \))
- \( X \) called reactance (\( \Omega \))
- \( G \) called conductance (\( S \))
- \( B \) called susceptance (\( S \))
Phasor Circuit Analysis Overview Continued

- Phasor circuit analysis is very similar to what we have already discussed.
- The only idea that is a little tricky is maximum power transfer.
- Everything that we learned earlier this term still applies.
- The next few lectures will consist of examples of how to apply.
- We have a few extra steps.

Example 3: Workspace

Find the equivalent input impedance.

Example 4: Equivalent Impedance

Find the equivalent input admittance.
Find the equivalent input impedance when the circuit is operating at a frequency of 1.6 Mrad/s.
Phasor Circuit Analysis Steps

1. Transform all independent sources to their phasor equivalent
2. Calculate the impedance ($Z$) of all passive circuit elements
3. Apply analysis methods that we learned earlier this term
4. Apply inverse phasor transform to obtain time-domain expression for currents and voltages of interest

Example 6: Equivalent Impedance

Find the equivalent impedance of a 10 µF capacitor in series with a 100 mH inductor when excited with a sinusoidal source operating at 1000 rad/s. Find the equivalent impedance when the capacitor is in parallel with the inductor. What are each of these equivalent to?

Example 7: Voltage Divider

$\begin{align*}
\text{For currents and voltages of interest:} \\
4. \text{ Apply inverse phasor transform to obtain time-domain expression for currents and voltages of interest.} \\
3. \text{ Apply analysis methods that we learned earlier this term.} \\
2. \text{ Calculate the impedance (Z) of all passive circuit elements} \\
1. \text{ Transform all independent sources to their phasor equivalent} \\
\end{align*}$

Example 6: Workspace
Use source transformations to solve for the steady-state part of $v_o(t)$.

The sinusoidal voltage sources are:

$v_1(t) = 240 \cos(4000t + 5.3^\circ) \, \text{V}$

$v_2(t) = 96 \sin(4000t) \, \text{V}$

Find the steady-state expression for $i_o(t)$ if $i_g(t) = 125 \cos(500t) \, \text{mA}$. 

\[ \Lambda (1000 + 24000 \cos(4000t + 5.3^\circ)) = (i)^2_a \]

\[ \Lambda (24000 \cos(4000t + 5.3^\circ) + 20) = (i)^1_a \]
Example 10: Kirchhoff's Laws

1. Find $I_a$, $I_c$, and $V_g$.

2. If $\omega = 800$ rads/s, write the expressions for $i_a(t)$, $i_c(t)$, and $V_g$.

The phasor current $I_b$ is $5\angle 45^\circ$ A.
Use the mesh-current method to find the branch currents $I_a$, $I_b$, $I_c$, and $I_d$.

Example 12: Mesh-Current Method

Use the node-voltage method to find the phasor voltage $V_o$.

Example 12: Node-Voltage Method
Find the Thévenin and Norton equivalents of the circuit in the phasor domain.
Use superposition to solve for the steady-state part of \( v_o(t) \). The sinusoidal voltage sources are:

\[
\begin{align*}
&v_1(t) = 240 \cos(2000t + 53.13) V \\
&v_2(t) = 96 \sin(8000t) V
\end{align*}
\]

Find the Thévenin and Norton equivalents of the circuit in the phasor domain.
Example 17: Operational Amplifiers

Find the steady-state expression for \( v_o(t) \) given that \( v_g(t) = 2 \cos(105t) \) V.

\[
\begin{align*}
\text{\( v_g(t) \)} &= 2 \cos(105t) \\
\text{\( v_o(t) \)} &= (t)^{0.5}
\end{align*}
\]
Example 18: Workspace (2)

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Example 17: Workspace (1)

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Example 18: Operational Amplifiers

0.1 nF

v

o

v

g

20 k

80 k

160 k

200 k

Find the steady-state expression for $v_o(t)$ when $v_g(t) = 20 \cos(106t) \ V$. 

Example 18: Operational Amplifiers (2)