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Mass Balance Problems

Gerald Recktenwald
gerry@me.pdx.edu

1 Goal: An Analytical Model of Salinity Balance

These notes introduce methods of analysis for mixing of batches and flows of material. Mixing problems arise in applications of chemical, civil and mechanical engineering. For the fish tank problem, these analytical tools enable us to predict the response of the fish tank to additions of salty and fresh water.

1.1 Learning Objectives

After studying these notes you should be able to

1. Identify whether a process is batch, steady flow or unsteady flow.
2. Draw a schematic of a steady flow process given a verbal description of inputs and outputs.
3. Write the mass balance equations for batch and steady flow processes.
4. Use a systematic procedure to solve batch and steady-flow mass balance problems.

These notes are written with the assumption that you already understand

- how to work with units in engineering analysis.
- how to compute mass fractions.
- how to use algebra to solve for unknowns in a system of two equations.

1.2 Other references

Engineering models of mass balance appear in books on Thermodynamics and Chemical Engineering processes. An excellent free reference is the Wikibook, *Introduction to Chemical Engineering Processes*[1]

2 Types of Mass Balance Problems

We begin by describing different types of mass balance problems.

2.1 Common Themes and Definitions

Different mass balance problems share the same core idea that *mass is always conserved*. We will show three different types of problems: batch, steady flow and unsteady flow. The difference is in how the engineering analysis is performed.

Another common idea is that all problems involve a mixing of ingredients. We need to be able to define the constituents of each ingredient.

[1] The whole book is found at `http://en.wikibooks.org/wiki/Introduction_to_Chemical_Engineering_Processes`. A helpful starting point for the material discussed in these notes is the section *What is a mass balance?*, `http://en.wikibooks.org/wiki/Introduction_to_Chemical_Engineering_Processes/What_is_a_mass_balance%3F`.

Mole

A mole is 6.022×10^{23} unitary entities of a substance. The unitary entities could be molecules, atoms, ions, electrons or others. In other words, a mole is a number or count of individual things.

A number of unitary entities in a mole is called *Avogadro's number*.

$$\text{Avogadro's number} = 6.022 \times 10^{23}$$

Avogadro's number has no units since it is simply the number of things.

Mass Fraction

Mass fraction is the mass ratio of a constituent substance to the total mass of a mixture of constituents. The symbol for mass fraction is capital X. The constituent is identified by a subscript. Thus, the mass fraction of constituent A in a mixture is

$$X_A = \frac{\text{mass of } A}{\text{mass of the mixture}}$$

Note that mass fractions of *all* constituents in a mixture must add to one.

$$X_A + X_B + X_C + \cdots = 1$$

For example in a saltwater mixture of 0.05 wt % salt we know that

$$X_{\text{salt}} + X_{\text{water}} = 1$$

Therefore, since $X_{\text{salt}} = \dfrac{0.05}{100} = 0.0005$, we then know that $X_{\text{water}} = 1 - X_{\text{salt}} = 0.9995$

2.2 Batch Processes

A batch process is a simple model of mixing. Figure 1 shows schematic representations of batch systems. Batch processes have these characteristics:

- In batch processes, there is a *initial* state, typically with separate constituents, and an *final* state, where the constituents are mixed.
- In batch processes, there is no steady flow of materials.

We can reverse the order of the process where the final state has separate constituents. Sometimes, problems with steady flows can be converted to batch problems. However, if there is no flow, then use a batch model in your analysis.

2.3 Steady flow Problems

Steady flow problems involve streams of material entering and leaving a mixing region. Figure 2 is a schematic of a steady flow mixing process involving two incoming streams and two outgoing streams. In general, there is no limit on the number of streams, but in typical systems the total number of streams is small, say 2 to 5. The total mass flow rate in each stream is designated by $\dot{m}$ and a subscript to identify the stream.

In a steady flow problem, the flow rates are steady, i.e., they do not change with time. As a consequence, the mass in the mixing region is also steady.

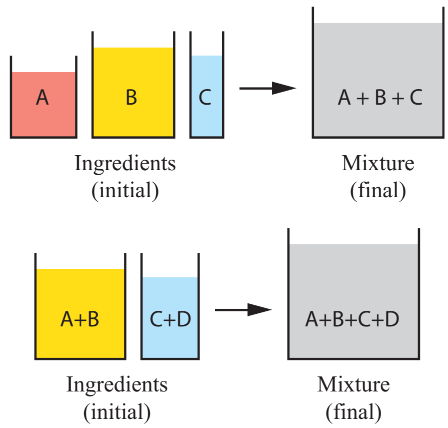


Figure 1: Schematic of batch mixing processes. Constituents A, B, C and D may or may not undergo chemical interactions.

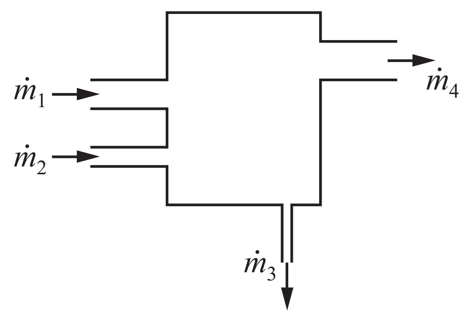


Figure 2: Steady flow model of a mixing process.

Steady flow problems involve flow *rates*, meaning that the material is continuously crossing the boundary of the mixing region. There is no *initial* state and *final* state because the system experiences a continuous process. If we were to attach sensors to the input and output streams (say temperature or pressure or salinity sensors), these sensors would have different values depending on their location in the streams, but the output from the sensors would be constant because the entire system is steady.

2.4 Unsteady Flow Problems

In an unsteady flow problem the streams entering and leaving the mixing region can change with time. The material in the mixing region can also accumulate (or decrease) with time. Unsteady flow problems are more complex than steady flow problems. We will stick to steady flow problems in this course.

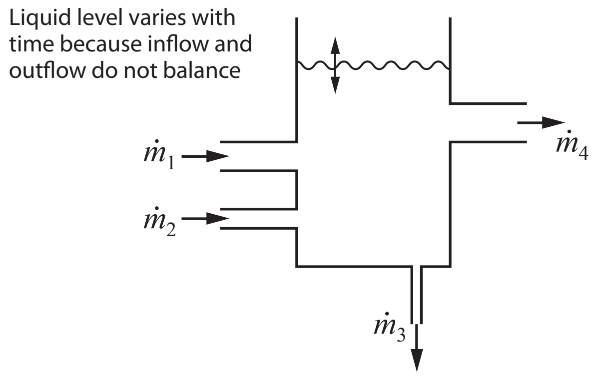


Figure 3: Unsteady flow model of a mixing process.

2.5 Flow Rate Definitions

Mass flow rate is a quantity indicating the amount of mass crossing a defined boundary per unit time. The symbol for mass flow rate is $\dot{m}$,

$$\dot{m} = \frac{\text{mass crossing a boundary}}{\text{period of time}}.$$

The amount of material crossing a boundary can also be expressed as a volumetric flow rate. The volumetric flow rate is designated Q,

$$Q = \frac{\text{volume crossing a boundary}}{\text{period of time}}.$$

Note that the symbol for volumetric flow rate, Q, is also used in heat transfer to denote a rate of heat flow[2].

We can get a physical feel for volumetric and mass flow rates by considering the filling of a bucket from a faucet as depicted in Figure 4. If the flow rate is steady, then

$$Q = \frac{\Delta V}{t_2 - t_1} = \frac{\Delta V}{\Delta t} \tag{1}$$

where ΔV is the change in volume during the time interval $t_2 - t_1 = \Delta t$, and v is the downward fluid velocity.

The mass flow rate could be obtained if a weighing scale was placed under the bucket in Figure 4. We would then measure the change in mass of water

$$\dot{m} = \frac{\Delta m}{t_2 - t_1} = \frac{\Delta V}{\Delta t} \tag{2}$$

where Δm is the change in mass during the time interval Δt.

In practical engineering problems we cannot usually measure flow rates with buckets. One of the primary goals of using mass balances is to reduce the number of measurements by relating flow rates to each other. We also use fluid property definitions. Mass and volume are related by the density

$$\rho = \frac{\text{mass of material}}{\text{volume of material}} \tag{3}$$

[2]There are only 26 letters in the roman alphabet, so some limits on the choice of symbols is inevitable. Within sub-disciplines the symbols are consistent. Across sub-disciplines the symbols get reused.

For many common liquids and gases, the density is tabulated or available in formulas called equations of state. For our purposes, we will assume that the density is known.

From the definition of density in Equation (3), the relationship between mass and volume, as well as the relationship between mass flow rate and volumetric flow rate is

$$\begin{array}{ccccc} \text{mass} & = & \text{density} & \times & \text{volume} \\ m & = & \rho & & V \end{array}$$

$$\begin{array}{ccccc} \begin{array}{c}\text{mass}\\ \text{flow rate}\end{array} & = & \text{density} & \times & \begin{array}{c}\text{volumetric}\\ \text{flow rate}\end{array} \\ \dot{m} & = & \rho & & Q \end{array}$$

where V is the volume of the sample having mass m. Note that the flow rate can be steady, or it can change with time.

When considering flow in pipes, it is very helpful to relate the flow rates to the velocity of the fluid in the pipe. Figure 5 depicts the flow of water from a round pipe into a bucket. In a short time interval Δt, a slug of water in the pipe moves a distance $L = v\Delta t$, where v is the velocity of water in the pipe. The volume of water in that slug is

$$\Delta V = LA = (v\Delta t)A$$

where A is the cross-sectional area of the duct. Think of LA as the volume swept by the moving slug of water in Δt. Dividing both sides of the preceding equation by Δt gives

$$\boxed{Q = vA} \tag{4}$$

and since $\dot{m} = \rho Q$, Equation (4) gives

$$\boxed{\dot{m} = \rho v A} \tag{5}$$

Equation (4) and (5) are two fundamental equations for the volumetric and mass flow rate through a pipe or duct.

2.6 Mass Balance Equation for the System

The *Conservation Principle* is that the physical properties of material in a system is completely accounted for by three processes: (1) the net inflow or outflow of material across the system boundaries, (2) the generation or consumption of material in the system by chemical reactions, or (3) a

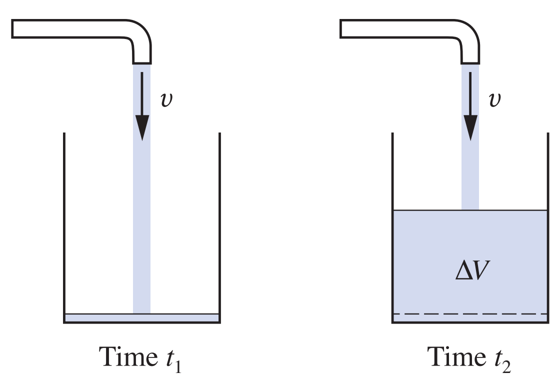


Figure 4: Volumetric flow rate during the filling of a bucket. The flow rate is assumed to be steady.

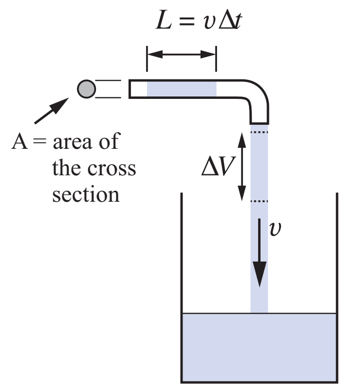


Figure 5: Volumetric flow rate related to the velocity of flow in a pipe.

change in the storage of material in the system. By *material* we can mean either the total mass in the system or individual chemical species. The conservation principle can be expressed in the following quasi-mathematical expression.

$$\begin{matrix}\text{Rate of}\\\text{accumulation}\end{matrix} = \begin{matrix}\text{Rate of}\\\text{inflow}\end{matrix} - \begin{matrix}\text{Rate of}\\\text{outflow}\end{matrix} + \begin{matrix}\text{Rate of}\\\text{generation}\end{matrix} - \begin{matrix}\text{Rate of}\\\text{consumption}\end{matrix} \tag{6}$$

This generic equation will be applied to the conservation of overall mass, and to the conservation of individual chemical species.

2.6.1 Overall Mass Balance

Mass is not created or destroyed by chemical reactions. Therefore, applying the conservation principle in Equation (6) to the mass in a system gives

$$\begin{matrix}\text{Rate of}\\\text{accumulation}\\\text{of mass}\end{matrix} = \begin{matrix}\text{Rate of}\\\text{inflow}\\\text{of mass}\end{matrix} - \begin{matrix}\text{Rate of}\\\text{outflow}\\\text{of mass}\end{matrix}. \tag{7}$$

Equation (7) *always* applies. We need to be more specific about each of the terms before this equation is useful in an engineering analysis. In most situations this general formula reduces to a much simpler form.

2.6.2 Mass Balance for constituents

Chemical reactions generally involve the combination of one set of chemical species (e.g., molecules, atoms, ions, etc.) to form other set of chemical species. For a general, unsteady system as depicted in Figure 3, the conservation of species k requires

$$\begin{matrix}\text{Rate of}\\\text{accumulation}\\\text{of } k\end{matrix} = \begin{matrix}\text{Rate of}\\\text{inflow}\\\text{of } k\end{matrix} - \begin{matrix}\text{Rate of}\\\text{outflow}\\\text{of } k\end{matrix} + \begin{matrix}\text{Rate of}\\\text{generation}\\\text{of } k\end{matrix} - \begin{matrix}\text{Rate of}\\\text{consumption}\\\text{of } k\end{matrix} \tag{8}$$

Equation (8) *always* applies, but it is not particularly useful in this form except as a conceptual statement about the processes that contribute to conservation of species.

3 Analysis of Mass Balance Problems

We now turn to use basic definitions and conservation principles in practical problems.

3.1 Analysis of Batch Problems

Batch problems have an identifiable initial and final state. Batch problems do not involve flow. Since there is no inflow or outflow, the rate of accumulation (or loss) of mass inside the system must be zero. Therefore, for batch problems the mass conservation principle can be reduced to

$$\text{Initial mass} = \text{Final mass} \tag{9}$$

To use this principle to set up and solve problems, we need to translate the words into a form that can be used in equations. We use m to denote mass, and A, B, C, etc. to identify constituents. The mass conservation principle applies to the total mass in the system, and to each of the constituents.

$$\begin{aligned}
m_{\text{initial}} &= m_{\text{final}} && \text{total mass} \\
m_{A,\text{initial}} &= m_{A,\text{final}} && \text{constituent A} \\
m_{B,\text{initial}} &= m_{B,\text{final}} && \text{constituent B} \\
m_{C,\text{initial}} &= m_{C,\text{final}} && \text{constituent C} \\
&\vdots
\end{aligned}$$

It is clumsy to write "initial" and "final" in mathematical expressions, so we will use "i" for initial and "f" for final. We also use the convention that if no constituent (A,B,C,...) is identified, then the mass is the total system mass. Thus, the mass balance principle for batch process can be written.

$$\begin{aligned}
m_i &= m_f \\
m_{A,i} &= m_{A,f} \\
m_{B,i} &= m_{B,f} \\
m_{C,i} &= m_{C,f}
\end{aligned}$$

Stepwise Procedure for Batch Problems

The following steps are used to solve a batch problems.

1. Identify the problem as suitable for a batch model.
2. Make a sketch of the system, and identify the initial state and the final state.
3. Write down the expressions for the total mass in the initial and final state. Write down the expressions for the mass of constituents (or moles of chemical species) in the initial and final state.
4. Set masses at initial state equal to masses at the final state, i.e., invoke the mass conservation principle.
5. Identify which of the quantities of mass and species are known and which are unknown.
6. Use algebra to solve for the unknowns.
7. Check your work for consistency.

The first step will be significant when looking at new problems, where you also have the options of using a steady flow or an unsteady flow model.

The stepwise procedure may seem confining or pedantic. However, as mixing problems become more complex, the systematic approach provides an organized way to set up and solve problems. Stepwise procedures help solve problems that may seem impossible at first.

Regardless of the procedure,

Always check your units!

Example 1 Adjusting Salinity of a Saltwater Mixture

Problem Statement:

A 10-gallon aquarium contains 2 percent salt by weight. How much salt would you need to add to bring the salt concentration to 3.5 percent salt by weight?

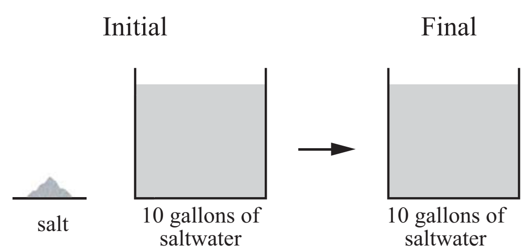
Solution:

Step 1: Determine whether this is a batch, steady flow or unsteady flow problem.
We are given a known initial state and final state.

Initial:	Unknown amount of salt, and a mixture with known volume and mass fraction of salt
Final:	Mixture with desired (known) mass fraction of salt and unknown total mass of the mixture.

There is no flow. Therefore, treat this as a *batch* problem.

Step 2: Make a sketch, and identify initial and final state.



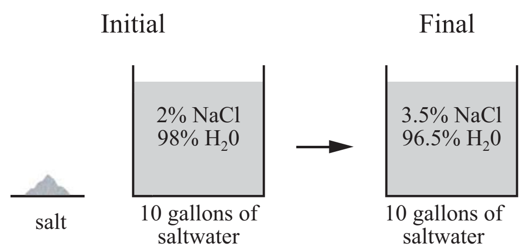
Use these symbols

m_s	mass of salt
m_{mix}	mass of saltwater mixture
X_s	mass fraction of salt
X_w	mass fraction of water

We know the initial and final mass fractions (or weight percents) of the salt and water

$$\begin{array}{lll} \text{Initial:} & X_s = 0.02 & X_w = 0.98 \\ \text{Final:} & X_s = 0.035 & X_w = 0.965 \end{array}$$

The mass fraction of the water is known because the mass fractions must add to one (refer to the definition of mass fraction on page 2). With that information, we can re-draw the system.



We also know the following physical parameters and conversion factors.

$$1 \text{ gallon} = 0.1337\,\text{ft}^3, \qquad \text{weight density of water} = \rho g = 62.3\,\frac{\text{lb}_\text{f}}{\text{ft}^3}$$

From this we can compute the initial weight of the water

$$W_w = \rho g V = 62.3\,\frac{\text{lb}_\text{f}}{\text{ft}^3} \times 10\,\text{gal} \times \frac{0.1337\,\text{ft}^3}{1\,\text{gal}} = 83.3\,\text{lb}_\text{f}$$

Note that the data is given in form of weights not masses, so we will eventually multiply our mass conservation equations by g, the acceleration of gravity, to convert mass to weight.

Step 3: Write expressions for mass and species initial and final state.
Using symbols for all quantities, we can write down the masses of the components before and after the mixing.

	Initial	Final
salt:	$m_{s,\text{add}} + X_{s,i} m_{\text{mix},i}$	$X_{s,f} m_{\text{mix},f}$
water:	$X_{w,i} m_{\text{mix},i}$	$X_{w,f} m_{\text{mix},f}$
total:	$m_{\text{mix},i} + m_{s,\text{add}}$	$m_{\text{mix},f} = ?$

where $m_{s,\text{add}}$ is the unknown amount of salt to be added, and m_{mix} is the mass of the saltwater mixture. This set of terms looks intimidating, but the mass fractions are all known. There are really just two unknowns, $m_{s,\text{add}}$ and $m_{\text{mix},f}$.

Step 4: Set masses at initial and final state equal

$$m_{s,\text{add}} + X_{s,i} m_{\text{mix},i} = X_{s,f} m_{\text{mix},f} \tag{10}$$
$$X_{w,i} m_{\text{mix},i} = X_{w,f} m_{\text{mix},f} \tag{11}$$
$$m_{\text{mix},i} + m_{s,\text{add}} = m_{\text{mix},f} \tag{12}$$

We are given values in weights, not masses, so multiply through by g, the acceleration of gravity

$$W_{s,\text{add}} + X_{s,i} W_{\text{mix},i} = X_{s,f} W_{\text{mix},f} \tag{13}$$
$$X_{w,i} W_{\text{mix},i} = X_{w,f} W_{\text{mix},f} \tag{14}$$
$$W_{\text{mix},i} + W_{s,\text{add}} = W_{\text{mix},f} \tag{15}$$

Step 5: Identify knowns and unknowns
These equations will look less intimidating when we substitute known values

$$W_{s,\text{add}} + (0.02)\,(83.3\,\text{lb}_\text{f}) = (0.035)\,W_{\text{mix},f} \tag{16}$$
$$(0.98)\,(83.3\,\text{lb}_\text{f}) = (0.965)\,W_{\text{mix},f} \tag{17}$$
$$83.3\,\text{lb}_\text{f} + W_{s,\text{add}} = W_{\text{mix},f} \tag{18}$$

This is a set of three equations in the two unknowns $W_{s,\text{add}}$ and $W_{\text{mix},f}$.

Step 6: Use algebra to solve for the unknowns
We can solve Equation (17) for $W_{\text{mix},f}$

$$\begin{aligned}(0.98)\,(83.3\,\text{lb}_\text{f}) &= (0.965)\,W_{\text{mix},f}\\ 81.63\,\text{lb}_\text{f} &= (0.965)\,W_{\text{mix},f}\\ W_{\text{mix},f} &= 84.59\,\text{lb}_\text{f}\\ &\boxed{W_{\text{mix},f} = 84.6\,\text{lb}_\text{f}}\end{aligned} \tag{19}$$

Now that $W_{\text{mix},f}$ is known, use either Equation (16) or Equation (18) to solve for $W_{s,\text{add}}$. We will use Equation (16) to solve for $W_{s,\text{add}}$, and save Equation (18) for a final check.

$$\begin{aligned}W_{s,\text{add}} + (0.02)\,(83.3\,\text{lb}_\text{f}) &= (0.035)\,W_{\text{mix},f}\\ W_{s,\text{add}} + 1.666\,\text{lb}_\text{f} &= (0.035)\,(84.59\,\text{lb}_\text{f})\\ W_{s,\text{add}} &= 2.961\,\text{lb}_\text{f} - 1.666\,\text{lb}_\text{f}\\ W_{s,\text{add}} &= 1.295\,\text{lb}_\text{f}\\ &\boxed{W_{s,\text{add}} = 1.30\,\text{lb}_\text{f}}\end{aligned} \tag{20}$$

Step 7: Check for consistency
Use Equation (18) to check the calculations

$$\begin{aligned}83.3\,\text{lb}_\text{f} + W_{s,\text{add}} &= W_{\text{mix},f}\\ 83.3\,\text{lb}_\text{f} + 1.30\,\text{lb}_\text{f} &\stackrel{?}{=} 84.6\,\text{lb}_\text{f}\\ 84.6\,\text{lb}_\text{f} &= 84.6\,\text{lb}_\text{f}\ \checkmark\end{aligned}$$

$\square$

3.2 Analysis of Steady Flow Problems

Steady flow systems can be identified by the following characteristics.

1. The system has continuous flow rates across its boundaries. Contrast with batch.
2. The flow rates do not change with time. Contrast with unsteady.
3. The generation and/or consumption of species does not change with time. Contrast with unsteady.

Figure 6 is a schematic of a steady flow system. The dashed line provides an imaginary boundary that separates the system, which is inside the boundary, from the environment.

For steady flow, there can be no accumulation or loss of mass inside the system. The mass conservation principle reduces to

$$\begin{matrix}\text{Rate of} \\ \text{inflow of mass}\end{matrix} = \begin{matrix}\text{Rate of} \\ \text{outflow of mass}\end{matrix} \tag{21}$$

In situations where there are several inflow and outflow boundaries, we can write Equation (21) with a summation notation

$$\sum_{\text{in}} \dot{m}_i = \sum_{\text{out}} \dot{m}_i \tag{22}$$

where the index i is interpreted as listing the numerical subscripts associated with either the inlet or outlet boundaries. Since the problem is steady, there is no subscript "i" for initial or subscript "f" for final.

The direction of the mass flow rates across the boundary is significant. Assuming that the direction of the arrows in Figure 6 are consistent with the actual flow directions across the boundary, the overall mass balance for the system depicted in Figure 6 is

$$\dot{m}_1 + \dot{m}_2 = \dot{m}_3 + \dot{m}_4$$

Mass flows crossing the boundaries can carry a different composition of constituents. The rate at which species A crossing boundary 1 is

$$\dot{m}_1 X_A$$

which would have units of kg_A/s in SI units.

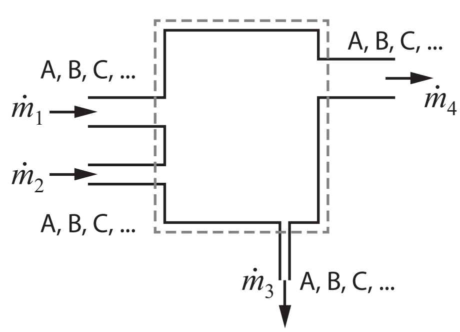


Figure 6: A control volume defines boundaries of the steady flow model of a mixing process.